1. (10 points) Let $U = \sqrt{Z_1^2 + Z_2^2}$ and $V = Z_1/Z_2$, where Z_1 and Z_2 are independent standard normal random variables. Find the joint density function of U and V. Are U and V independent?

2. (10 points) Let X and Y be random variables having the bivariate normal distribution with $E(X) = E(Y) = 0$, $Var(X) = Var(Y) = 1$ and $Cov(X, Y) = \rho$. Find the expected value of $\max\{X, Y\}$.

3. (10 points) Let $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ be the order statistics of a random sample from $U(\theta - \rho, \theta + \rho)$. Find the maximum likelihood estimators of θ and ρ. Are the maximum likelihood estimators unbiased?

4. (10 points) Let $X_1, X_2, \ldots, X_n$ be a random sample from a population with density function $f(x|\theta) = \frac{1}{2\theta}$, where $-\theta < x < \theta$ and $\theta > 0$. Find a best unbiased estimator of θ.

5. (10 points) Let $X_1, X_2, \ldots, X_n$ be a random sample from $N(0, \sigma^2)$. Consider the estimation of σ. Find the asymptotic relative efficiency between $T_1 = \sqrt{\pi/2}\sum_{i=1}^n |X_i|/n$ and $T_2 = \sqrt{\sum_{i=1}^n X_i^2/n}$.

6. Suppose that $X_1, X_2, \ldots, X_{15}$ is an iid sample from a distribution with pdf

$$f(x) = \frac{4}{\theta}x^3 e^{-x^4/\theta}, \ x > 0;$$

where $\theta > 0$ is an unknown parameter.

(a) (10 points) Use the Chi-Square distribution table below to find the rejection region for the most powerful (MP) test of $H_0 : \theta = 2$ against $H_1 : \theta = \theta_1$, $\theta_1 > 2$, at level $\alpha = 0.05$. (Hint: $X_i^4/\theta \sim Exponential(1)$)

(b) (5 points) If you observe $\sum_{i=1}^{15} x_i^4 = 46.98$, what is the p-value? What is your decision with the significance level of $\alpha = 0.05$?

(c) (5 points) What is the approximate power of the MP test at $\theta_1 = 5$?

(d) (10 points) Is the MP test also a uniformly most powerful (UMP) test for testing $H_0 : \theta = 2$ versus $H_1 : \theta > 2$? Explain it.

Table 1: Chi-Square Distribution Table

Degree of freedom	**Percentile**								
	0.034	0.05	0.1	0.25	0.75	0.9	0.95	0.975	0.99
15	6.68	7.26	8.55	11.04	19.31	22.31	25.00	27.49	30.58
30	17.51	18.49	20.60	24.48	34.80	40.26	43.77	46.98	50.89
40	25.31	26.51	29.05	33.66	47.27	51.81	55.76	59.34	63.69

7. (10 points) Let $X_1, X_2, \ldots, X_n$ be iid from a distribution with pdf

$$f(x) = \frac{2x}{\lambda}e^{-x^2/\lambda}, \ x > 0;$$

where λ and x are both positive. Find the level α UMP test for $H_0 : \lambda = 1$ versus $H_1 : \lambda > 1$.

8. (10 points) Let $X_1, X_2, \ldots, X_m$ be iid from a distribution with pdf

$$f(x) = \mu x^{\mu - 1},$$

where $\mu > 0$ and $0 < x < 1$. Let $Y_1, Y_2, \ldots, Y_n$ be iid from a distribution with pdf

$$g(y) = \theta y^{\theta - 1},$$

where $\theta > 0$ and $0 < y < 1$. Let two statistics

$$T_1 = \sum_{i=1}^m \log(X_i) \text{ and } T_2 = \sum_{j=1}^n \log(Y_j).$$

Find the likelihood ratio test statistic for $H_0 : \mu = \theta$ versus $H_1 : \mu \neq \theta$ in terms of T_1, T_2 and the MLEs.

試題隨卷繳回