Instructions:

- Answer all questions.
- Write the question number clearly on the answer sheet before writing the solution for each question.
- For fill-in-the-blank questions, only the final answers are graded.
- For other questions, show your reasoning and calculations. Presentation is also graded.
- No electronic device, computer algebra system or AI assistant is allowed for this exam.

1. (44 points) Fill in the blanks (4 points each).

(a) $\lim_{x\to 0^+} \frac{\sqrt{\sin x}}{\sqrt{x}} = \underline{(1a)}$　(b) $\frac{d}{dx}(2x)^{\pi} = \underline{(1b)}$　(c) $\frac{d}{dx}2^{-x} = \underline{(1c)}$

(d) $\frac{d^2}{dx^2}\sin^{-1} x = \underline{(1d)}$　(e) $\int \sec^2 x\, dx = \underline{(1e)}$　(f) $\frac{d}{dx}\int_1^{x^2} f\left(\sqrt{t}\right) dt = \underline{(1f)}$

(g) $\int_1^{\infty} xe^{-x}\, dx = \underline{(1g)}$　(h) $\lim_{x\to\infty}\left(\sqrt{x+\sqrt{x}}-\sqrt{x}\right) = \underline{(1h)}$

(i) $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\sqrt{n}\sqrt{n+k}} = \underline{(1i)}$　(j) $\frac{1}{(1-3x)^2} = \sum_{k=0}^{\infty}\underline{(1j)}\ x^k$ for $|x| < \underline{(1k)}$

2. Given the differential equation

$$(y-1)\frac{dy}{dx} + x(y-1)^2 = 0\,,$$

solve for $y = y(x)$ with the following initial conditions.

(a) (4 points) $y(0) = 1$;
(b) (8 points) $y(0) = 0$.

3. Evaluate the following integrals.

(a) (6 points) $\int_0^1\int_{\sqrt{y}}^1 e^{x^3}\, dxdy$.

(b) (6 points) $\iiint_P y^2\, dV$, where $P := \left\{(x,y,z)\in\mathbb{R}^3 \mid x^2+2y^2 \le z \le 2\right\}$.

4. Consider the function $f(x,y,z) = xy^2z^3$ defined on the region

$$Q := \left\{(x,y,z)\in\mathbb{R}^3 \mid x\ge 0\,,\, y\ge 0\,,\, z\ge 0\right\}\,.$$

(a) (4 points) Given a point $\mathbf{p} = (a,b,c)\in Q$ with $a>0$, $b>0$ and $c>0$, find the direction where f increases most rapidly at $\mathbf{p}$.

(b) (3 points) The minimum value of f on Q is $\underline{(4b)}$.

(c) (9 points) Find the maximum value of f on the surface

$$S := \left\{(x,y,z)\in Q \mid x^2+y^2+z^2 = 6\right\}\,,$$

i.e. the intersection of Q and the sphere of radius $\sqrt{6}$ centred at the origin.

5. Let

$$Q := \left\{(x,y,z)\in\mathbb{R}^3 \,\middle|\, 2z = 1-\sqrt{x^2+y^2}\,,\ z\ge 0\right\}\,,$$

which is a half cone in $\mathbb{R}^3$ and is endowed with the upward orientation (in the positive z-direction). Let

$$\mathbf{E} := \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

be a vector field on $\mathbb{R}^3$.

(a) (6 points) Find the divergence $\operatorname{div}\mathbf{E}$ of $\mathbf{E}$.

(b) (10 points) Compute the surface integral $\iint_Q \mathbf{E}\cdot d\mathbf{S}$ over Q.

(You may want to consider the hemisphere $S := \left\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2 = 1\,,\ z\ge 0\right\}$.)

試題隨卷繳回