

1. Evaluate

(a)  $\int \int_R e^{x+y} dx dy$ , where  $R = \{(x, y) \in \mathbb{R}^2: |x| + |y| \leq 1\}$  (5 points)

(b)  $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+xy+y^2)} dx dy$  (10 points)

2. Let  $f$  be a strictly monotonic function defined on the interval  $[a, b]$  and let  $g$  be its inverse function such that  $g(f(x)) = x$ . Assume that  $f'$  exists and is continuous on  $[a, b]$ .

(a) Show that  $\int_a^b f(x) dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} g(y) dy$  (10 points)

(b) Use (a) to evaluate  $\int_1^e \ln(x) dx$  (5 points)

3. An  $n \times n$  matrix  $P$  is an orthogonal projection matrix onto a subspace  $S = \text{Im}(P) \subseteq \mathbb{R}^n$  if, for every  $v \in \mathbb{R}^n$ , the residual vector  $(v - Pv)$  is orthogonal to  $S$ ; that is,  $w^T(v - Pv) = 0$  for all  $w \in S$ .

(a) Prove that  $P$  is an orthogonal projection matrix if and only if  $P$  is idempotent and symmetric. (10 points)

(b) Suppose  $A$  is an  $n \times p$  matrix with full column rank. Let  $P_A = A(A^T A)^{-1} A^T$ . Show that  $P_A$  is an orthogonal projection matrix and specify the subspace it projects onto. (5 points)

4. Let  $V$  be a vector space.

(a) Define what it means for a set of vectors  $\{v_1, \dots, v_k\} \subseteq V$  to be linearly independent. (5 points)

(b) Let  $T: V \rightarrow V$  be a linear transformation. Prove that, if  $v_1, \dots, v_k$  are eigenvectors of  $T$  corresponding to distinct eigenvalues  $\lambda_1, \dots, \lambda_k$ , then the set  $\{v_1, \dots, v_k\}$  is linearly independent. (10 points)

5. (a) Use the method of Lagrange multipliers to find the maximum value of the function:

$$f(x_1, \dots, x_n, y_1, \dots, y_n) = \sum_{i=1}^n x_i y_i$$

subject to the constraints  $\sum_{i=1}^n x_i^2 = 1$  and  $\sum_{i=1}^n y_i^2 = 1$ . (8 points)

(b) Use results from (a) to prove the Cauchy-Schwarz inequality:  $\sum_{i=1}^n a_i b_i \leq \sqrt{\sum_{i=1}^n a_i^2} \sqrt{\sum_{i=1}^n b_i^2}$ . (2 points)

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題號： 72  
科目：數學(B)  
節次： 7

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題號： 72  
共 2 頁之第 2 頁

6. Let

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

- (a) Find an eigen-decomposition of  $A$  of the form  $A = PDP^{-1}$ , where  $D$  is a diagonal matrix and  $P$  is an invertible matrix. (10 points)
- (b) Find a matrix  $B$  such that  $B^2 = A$ . (You may express your answer in terms of  $P$  and  $P^{-1}$ ) (5 points)
7. Let  $M_{n \times n}(\mathbb{R})$  denote the vector space of all  $n \times n$  real matrices. Recall that for any square matrix  $C \in M_{n \times n}(\mathbb{R})$ , the trace of  $C$  is defined by  $\text{tr}(C) = \sum_{i=1}^n c_{ii}$ , where  $c_{ii}$  denotes the  $i$ -th diagonal element of  $C$ .
- Define a mapping  $\langle \cdot, \cdot \rangle: M_{n \times n}(\mathbb{R}) \times M_{n \times n}(\mathbb{R}) \rightarrow \mathbb{R}$  by
- $$\langle A, B \rangle = \text{tr}(A^T B), \quad A, B \in M_{n \times n}(\mathbb{R}).$$
- Prove that this mapping defines an inner product on  $M_{n \times n}(\mathbb{R})$ . (15 points)
- (Hint: Verify that the mapping satisfies the four axioms of an inner product: positivity, symmetry, additivity, and homogeneity.)

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