- **The problems in this exam are not ordered by difficulty.**
- **The following multiple-choice problems each has exactly one correct answer. You lose no points for incorrect answers. No explanation is required.**

1. (2%) Let F be a vector space, which of the following is not a consequence? (a) If $a, b \in F$, then $a - b \in F$. (b) If $a \in F$, then $a + a = 2 \cdot a$. (c) If $a, b \in F$ then $a \cdot b \in F$. (d) If $a, b, c \in F$, then $(a + b) + c = a + (c + b)$.

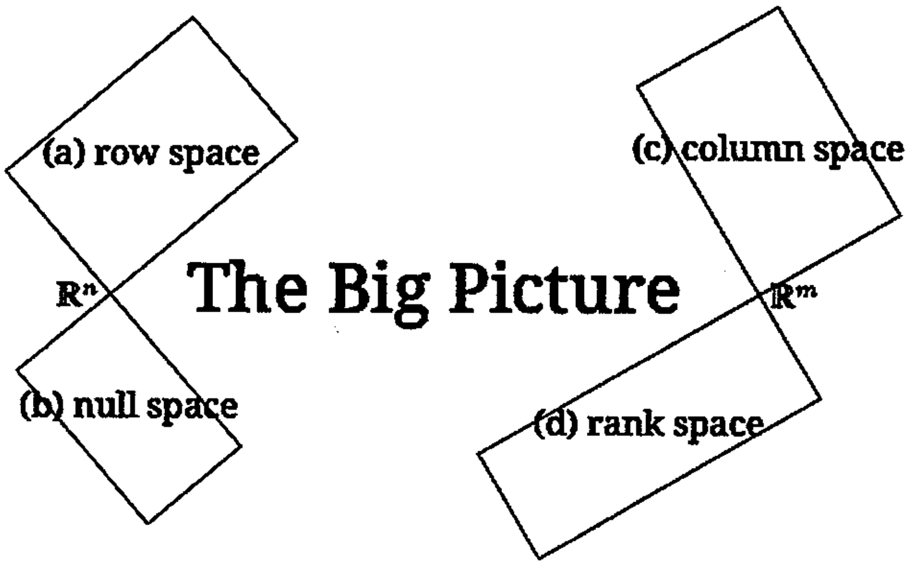
2. (2%) Which of the following is not able to serve as a vector space over the field of real numbers $\mathbb{R}$? (a) The complex numbers $\mathbb{C}$. (b) The set of all real numbers x such that $x^2 \leqslant 0$. (c) The functions $f : \mathbb{R} \to \mathbb{R}$ that are differentiable. (d) The lattice $\mathbb{Z}^n$ where n is an integer greater than 2026. (e) The set $\{x \in \mathbb{R} : x > 0\}$ where we use xy as x plus y.

3. (2%) According to the most strict definition of linear transformation, which of the following is not a linear transformation? (a) $L(x) = -x$. (b) $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. (c) The sum of another two linear transformations. (d) Mapping any vector x to the real number $0 \in \mathbb{C}$.

4. (2%) Which of the following can be used as the definition of the rank of a matrix M? (a) The width of M if M is invertible. (b) The dimension of the right null space $\{v : Mv = 0\}$. (c) The number of basis vectors of the column space of M. (d) The number of 1's in the reduced row echelon form of M.

5. (2%) Professor Gilbert Strang wrote the textbook *Introduction to Linear Algebra* for undergraduate students. The book is almost 600 pages long. Which of the following topics is on page 58? (a) Pseudo inverse. (b) Matrix multiplication. (c) Positive semi-definite. (d) Singular value decomposition.

6. (2%) On page 187 of *Introduction to Linear Algebra*, we can see a picture very similar to the following that explains the four fundamental subspaces:



Which of the (a), (b), (c), and (d) is labeled incorrectly?

7. (2%) Choose one incorrect statement regarding the augmented matrix $[A|b]$. (a) b is a column vector. (b) A is a matrix that is at least 2 columns wide. (c) It can be used to determine if the system $Ax = b$ has a solution. (d) It can be used to determine if the system $Ax = b$ has more than one solutions.

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8. (3%) How many 3×3 matrices are made of the numbers 0 and 1? (a) About 10. (b) About 20. (c) About 50. (d) About 100. (e) About 200. (f) About 500. (g) About 1000.

9. (3%) What is the determinant of the following matrix?

$$\begin{bmatrix} 0 & 0 & 55 & 4 & 0 \\ 7 & 0 & 51 & 90 & 0 \\ 67 & 0 & 57 & 85 & 19 \\ 32 & 84 & 72 & 34 & 11 \\ 0 & 0 & 63 & 92 & 0 \end{bmatrix}$$

(a) About 10^7. (b) About $2 \cdot 10^7$. (c) About $5 \cdot 10^7$. (d) About 10^8. (e) About $2 \cdot 10^8$. (f) About $5 \cdot 10^8$. (g) About 10^9.

10. (3%) Let Ψ be a square matrix. Let ψ_{ij} be the entry on the ith row and jth column of Ψ. Let ρ_i be the Gershgorin radius $\sum_{j \neq i} |\psi_{ij}|$. Let Δ_i be the Gershgorin disk $\{z \in \mathbf{C} : |z - \psi_{ii}| \leqslant \rho_i\}$. The Gershgorin circle theorem states that every eigenvalue of Ψ lies inside at least one of the Gershgorin disks. Consider the matrix

$$\begin{bmatrix} -10 & -1 & -3 & 2 \\ 0 & 30 & -3 & 0 \\ -1 & 3 & 20 & 0 \\ 0 & 3 & 0 & -40 \end{bmatrix}$$

Accordingly, which of the following is not an eigenvalue of the matrix? (a) -40.0001427912351045 (b) -11.1008990913169226 (c) 21.1196676389242784 (d) 28.9813742436277487

11. (3%) The power method refers to the following iterative process to find the dominant eigenvalue of a matrix T: Let u be an initial guess. Then repeat the following steps multiple times:

- $v = uT$,
- $u = v/v_1$, where v_1 is the first component of v.

Eventually, v_1 will be very close to the dominant eigenvalue of T. Now consider the matrix

$$T = \begin{bmatrix} 0 & 0 & 0 & 2i \\ 2i & 0 & 0 & 2i \\ 0 & 2i & 0 & 2i \\ 0 & 0 & 2i & 2i \end{bmatrix}$$

It is known that the dominant eigenvalue of T is 3.8[redacted]512395096585i. What is the missing digit? (a) 0 (b) 1 (c) 2 (d) 3 (e) 4 (f) 5 (g) 6 (h) 7 (i) 8 (j) 9 For this problem, we recommend using $u = [1, 2, 4, 8]$ as the initial guess and repeat the steps 10 times.

12. (3%) Jensen is playing with orthogonal matrices, that is, matrices O such that $O^*O = I$. He wants to characterize all real matrices K such that, for a very small real number ε, the matrix $O + \varepsilon OK$ is still orthogonal up to small error terms. More precisely, he wants that

$$(O + \varepsilon OK)^*(O + \varepsilon OK) - I$$

is a matrix whose entries are all multiples of ε^2 with no ε^1 or ε^0 terms. Which of the following conditions should K satisfy? (a) K is orthogonal. (b) K is orthonormal. (c) K is vertible. (d) K is invertible. (e) K is symmetric. (f) K is skew-symmetric.

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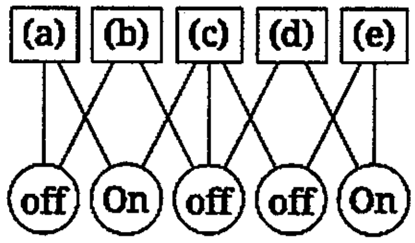
- **The following multiple-choice problems each has at least one correct answer, and it is possible that all the choices are correct.**

No partial credit will be given. You lose no points for incorrect answers. No explanation is required.

13. (3%) *Lights Out* is an electronic game where your goal is to turn off all the lights by pressing switches. In the following diagram, switches are represented by rectangles and lights are represented by circles. Pressing a switch toggles the on/off state of the lights connected to the switch.



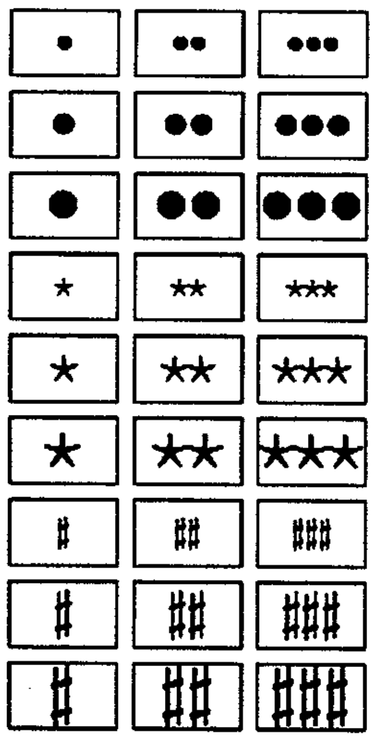
Which set of switches turn off all the lights if each is pressed once?

14. (3%) A binary linear block code C is generated by the following matrix

$$G = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

The code C is said to be *maximum distance separable* (MDS) if all 3×3 minors of G are odd numbers. However, C is not MDS. Which three columns of G give rise to an even minor? (a) $\{1,2,3\}$ (b) $\{1,2,4\}$ (c) $\{1,2,5\}$ (d) $\{1,2,6\}$ (e) $\{1,3,4\}$ (f) $\{1,3,5\}$ (g) $\{1,3,6\}$ (h) $\{1,4,5\}$ (i) $\{1,4,6\}$ (j) $\{1,5,6\}$ (k) $\{2,3,4\}$ (l) $\{2,3,5\}$ (m) $\{2,3,6\}$ (n) $\{2,4,5\}$ (o) $\{2,4,6\}$ (p) $\{2,5,6\}$ (q) $\{3,4,5\}$ (r) $\{3,4,6\}$ (s) $\{3,5,6\}$ (t) $\{4,5,6\}$

15. (3%) The board game *Set* uses a deck of 27 cards. Each card has three attributes:



- Its shape can be •, ⋆, or ♯.
- Its size can be small, medium, or large.
- Its number can be one, two, or three.

A *triple* in this game is three cards such that

- either all shapes are the same or all different,
- either all sizes are the same or all different, and
- either all numbers are the same or all different.

For instance,  is a triple because all shapes are different, all sizes are the same (small), and all numbers are different. A subset of cards is said to be *triple-fruitful* if for any two cards in the subset, the third card that forms a triple with them is also in the subset. What are the possible sizes of a triple-fruitful subset of cards? (a) 1 (b) 3 (c) 5 (d) 7 (e) 9 (f) 11 (g) 13 (h) 15 (i) 17 (j) 19 (k) 21 (l) 23 (m) 25 (n) 27 (o) 29

• The following reordering problems ask you to reorder all the choices. All choices must be used. No partial credit will be given. You lose no points for incorrect answers. No explanation is required.

16. (4%) Let S be a symmetric matrix with real entries. Reorder the following choices to build a proof of the property that S has real eigenvalues. (a) Since S is symmetric, $L = U^*SU = U^*S^*U = (U^*SU)^* = L^*$. (b) Since L is lower-triangular as well as upper-triangular, it must be diagonal. (c) Let ULU^* be the Schur decomposition of S, where L is lower triangular and U is unitary. (d) Therefore, the diagonals of L are the eigenvalues of S, and they are equal to their complex conjugates.

17. (4%) Recall that, in Gaussian elimination, we find a row with a nonzero entry on the first column, move this row to the top, and use this row to eliminate all the other entries in the first column. We then ignore the first row and the first column and repeat the same process. In numerical linear algebra, we often do *pivoting* to reduce rounding errors. That is, when choosing the next row to move to the top, we choose the row with the largest (after taking absolute value) entry on the first column. Let's apply pivoting to the following matrix:

$$\begin{matrix}\text{(a)}\\ \text{(b)}\\ \text{(c)}\\ \text{(d)}\end{matrix}\begin{bmatrix}-3 & 9 & -2 & 4\\ 3 & -3 & 6 & 2\\ 6 & 2 & 4 & 2\\ -3 & 1 & 6 & 4\end{bmatrix}$$

Reorder the choices above to indicate their order of being selected as the row that maximizes the absolute value of the pivot.

• The following problem requires you to find expressions. You get one point for each correct expression. You lose no points for incorrect expressions. No explanation is required.

18. (4%) Strassen is a famous algorithm for matrix multiplication. Let A and B be two 2×2 matrices. And let C be AB, that is,

$$C_{11} = A_{11} \times B_{11} + A_{12} \times B_{21}, \qquad C_{12} = A_{11} \times B_{12} + A_{12} \times B_{22},$$
$$C_{21} = A_{21} \times B_{11} + A_{22} \times B_{21}, \qquad C_{22} = A_{21} \times B_{12} + A_{22} \times B_{22}.$$

Note that the preceding equations require eight multiplications. Strassen, instead, recommends computing the following seven products:

$$M_1 = (A_{11} + A_{22}) \times (B_{11} + B_{22}), \qquad M_2 = (A_{21} + A_{22}) \times B_{11},$$
$$M_3 = A_{11} \times (B_{12} - B_{22}), \qquad M_4 = A_{22} \times (B_{21} - B_{11}),$$
$$M_5 = (A_{11} + A_{12}) \times B_{22}, \qquad M_6 = (A_{21} - A_{11}) \times (B_{11} + B_{12}),$$
$$M_7 = (A_{12} - A_{22}) \times (B_{21} + B_{22}).$$

(a) Express C_{11} as a linear combination of $M_1, M_2, \ldots, M_7$.
(b) Express C_{12} as a linear combination of $M_1, M_2, \ldots, M_7$.
(c) Express C_{21} as a linear combination of $M_1, M_2, \ldots, M_7$.
(d) Express C_{22} as a linear combination of $M_1, M_2, \ldots, M_7$.

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• For the following problems, you need to provide detailed derivation to each problem to get its full score.

19. Denote the bus waiting time by a continuous random variable X. On a rainy day, the bus waiting time is uniformly distributed on the interval $[2, 7]$. On a non-rainy day, the bus waiting time is uniformly distributed on the interval $[2, 12]$. The probability that a given day is rainy is 0.4.
(a) (5%) Derive the PDF $f_X(x)$. (Please express your answer in fractional form for full credit.)

(b) (5%) Compute $\mathbf{E}[X]$.
(c) (5%) Compute $\mathrm{Var}(X)$. (Please express your answer in fractional form for full credit.)

20. (5%) Y is a Poisson random variable with mean 5. Please compute $\mathbf{E}[3^Y]$.

21. The joint PMF of M and J is given as

$$P_{M,J}(m, j) = \frac{0.25\,(0.75)^{m-1}}{m},$$

for $m = 1, 2, \ldots$ and $j = 1, \ldots, m$. Otherwise, $P_{M,J}(m, j) = 0$. Let A denote the event that $M \geq 5$.
(a) (5%) Derive the marginal PMF $P_M(m)$.
(b) (5%) Derive the conditional PMF $P_{M|A}(m) = \mathbf{P}(M = m \mid A)$.
(c) (5%) Derive $\mathbf{E}[M \mid A]$.

22. (5%) Mike and Tom have an appointment this afternoon. Mike arrives uniformly at random between 2:00 PM and 3:00 PM, and Tom arrives uniformly at random between 2:30 PM and 3:30 PM. Assume their arrival times are independent. The person who arrives first will wait for at most 15 minutes for the other person before leaving. Find the probability that Mike and Tom meet (i.e., that they are both present at the same time). You can express your answer in fractional form.

23. Let T_1, T_2, T_3 be i.i.d. exponential random variables with mean 2. Let

$$U = T_1 + 2T_2 + 3T_3.$$

(a) (5%) Derive the MGF

$$\phi_U(s) = \mathbf{E}\left[e^{sU}\right],$$

and state the valid range of s.
(b) (5%) Compute $\mathrm{Var}(U)$.