

Problem 1. Multiple Choice (25 points)

A single facility operates a machine that serves arriving customers. At the start of each day d , the machine state is observed as $X_d \in \{G, B\}$ (Good/Bad). System manager will choose one action for that day based on X_d : Action C = "continue the operation"; Action M = "perform maintenance". If action C is chosen, the next-day state follows the following probability transitions:

$$P(G \rightarrow G) = 0.8, P(G \rightarrow B) = 0.2, P(B \rightarrow G) = 0.3, P(B \rightarrow B) = 0.7,$$

and the day's action cost is

$$\text{cost}(G, C) = 1 \text{ and } \text{cost}(B, C) = 6.$$

If action M is chosen, the next-day state of the machine is "Good" with probability one, and the action cost is

$$\text{cost}(G, M) = \text{cost}(B, M) = 4.$$

There are two stationary decision policies that choose an action based on the current machine state:

π_1 : choose action C if $X_d = G$ or $X_d = B$; that is, $\pi_1(G) = \pi_1(B) = C$.

π_2 : choose action C if $X_d = G$, choose action M if $X_d = B$; i.e., $\pi_2(G) = C$ and $\pi_2(B) = M$.

(1) (5 points) What is the induced probability transition matrix under the stationary policy π_2 ?

(A) $\begin{pmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{pmatrix}$ (B) $\begin{pmatrix} 0.8 & 0.2 \\ 1.0 & 0.0 \end{pmatrix}$ (C) $\begin{pmatrix} 1.0 & 0.0 \\ 0.3 & 0.7 \end{pmatrix}$ (D) $\begin{pmatrix} 1.0 & 0.0 \\ 1.0 & 0.0 \end{pmatrix}$ (E) $\begin{pmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{pmatrix}$

(2) (5 points) Under the stationary policy π_1 , what is the conditional probability $P(X_{d+2} = G | X_d = G)$?

(A) 0.58 (B) 0.64 (C) 0.70 (D) 0.76 (E) 0.8

(3) (5 points) Let $\bar{c}(\pi)$ be the long-run average daily cost under policy π . Which statement is correct?

(A) $\bar{c}(\pi_1) = 3.0, \bar{c}(\pi_2) = 1.5; \pi_2$ is preferred (B) $\bar{c}(\pi_1) = 1.5, \bar{c}(\pi_2) = 3.0; \pi_1$ is preferred
(C) $\bar{c}(\pi_1) = 2.0, \bar{c}(\pi_2) = 1.5; \pi_2$ is preferred (D) $\bar{c}(\pi_1) = 3.0, \bar{c}(\pi_2) = 2.0; \pi_2$ is preferred
(E) $\bar{c}(\pi_1) = 2.5, \bar{c}(\pi_2) = 1.5; \pi_2$ is preferred

During each day, assume customers arrive at rate $\lambda = 10/\text{hour}$ and are served in first-come-first-serve (FCFS) scheme, where service times are i.i.d exponential with mean = 5 minutes.

(4) (5 points) Assume the service system is described by a simple M/M/1 queue, what is the probability that the system is empty when a customer arrives?

(A) 1/12 (B) 1/10 (C) 1/6 (D) 1/2 (E) 5/6

(5) (5 points) Denote the hourly service rate of this M/M/1 system by μ , the expected number of customers in the system by L , and the expected time staying in the system by W . A steady state of the system requires $\lambda < \mu$. However, suppose demand increases so that $(\lambda/\mu) \rightarrow 1^-$, which of the following statements is correct?

(A) L and W both stay bounded (B) $L \rightarrow \infty$ and $W \rightarrow \infty$ (C) $L \rightarrow 0$ and $W \rightarrow 0$
(D) L stays bounded and $W \rightarrow \infty$ (E) $L \rightarrow \infty$ and W stays bounded

Problem 2 (25 points). A firm chooses production levels $x_1 \geq 0$ and $x_2 \geq 0$ to maximize profit. Revenue exhibits diminishing returns

$$R(x_1, x_2) = 20 \ln(1 + x_1) + 12 \ln(1 + x_2),$$

and the operating cost is

$$C(x_1, x_2) = 2x_1 + 3x_2.$$

Two shared resources impose two constraints:

$$x_1 + 2x_2 \leq 12 \text{ and } 2x_1 + x_2 \leq 10.$$

Denote the profit by $f(x_1, x_2) = R(x_1, x_2) - C(x_1, x_2)$, consider the optimization problem:

$$\max_{x_1, x_2} f(x_1, x_2) \text{ s.t. the constraints above.}$$

(1) (7 points) Show that this is a convex-optimization problem in the sense that (i) $f(x_1, x_2)$ is a concave function; and (ii) the feasible set is convex.

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- (2) (10 points) Write the Lagrangian and the complete KKT conditions (primal feasibility, dual feasibility, complementary slackness, stationarity). Use multipliers $\lambda_1, \lambda_2 \geq 0$ for the two resource constraints and $\mu_1, \mu_2 \geq 0$ for $x_1 \geq 0, x_2 \geq 0$.
- (3) (8 points) Since the solution obtained from unconstrained maximizer is infeasible, let us assume the optimum satisfies: $2x_1 + x_2 = 10$ (binding), $x_1 + 2x_2 < 12$ (slack), and $x_1 > 0, x_2 > 0$. Find a feasible optimum solution (x_1^*, x_2^*) .

Problem 3 (30 points). Consider the following linear programming (LP) problem:

$$\begin{aligned} \max & 3x_1 + 4x_2 + 10x_3 \\ \text{s.t.} & x_1 + 2x_2 + 3x_3 \leq 10, \quad 5x_1 + x_2 + 4x_3 \leq 6, \quad x_1, x_2, x_3 \geq 0 \end{aligned}$$

- (1) (5 points) In the first iteration of the simplex method, let us choose (x_1, x_2, x_3) as the non-basic variables. If we further choose x_1 to be the entering-basis variable, what would be the improvement in the objective value after this iteration?
- (2) (5 points) Using the same (x_1, x_2, x_3) as the non-basic variables. What would be the improvements in the objective value if we choose x_2 or x_3 to be the entering-basis variable, respectively?
- (3) (5 points) Compare the improvements in the objective value resulting from choosing either x_1, x_2 , or x_3 as the entering-basis variable (the values you calculated in the first two questions). Complete this iteration of simplex method by choosing the entering-basis variable with the "largest" improvement (we shall call this the "maximum improvement pivot rule").
- (4) (10 points) Using the maximum improvement pivot rule to complete the rest of the simplex method. Clearly write out the optimal solution (x_1^*, x_2^*, x_3^*) as well as the optimal objective value z^* .
- (5) (5 points) Let (y_1, y_2) denote the dual variables for the dual of this problem. What is the dual optimal solution (y_1^*, y_2^*) and the corresponding dual optimal objective

Problem 4 (20 points). Consider the following knapsack problem. You have a bag to pack up to three items you want, and they worth 3, 5, and 2 dollars respectively. Moreover, each of the item weighs 2, 4, and 1 kilo-grams respectively, and you can only bear no more than 5 kilograms in your bag. The parameters of this problem are summarized in the next table.

	Item 1	Item 2	Item 3
Value (dollars)	3	5	2
Weight (kg)	2	4	1

- (1) (5 points) You want to decide which item(s) to put in your bag in order to maximize the total value while obeying the weight constraint. Formulate this problem into a binary integer programming (BIP) problem.
- (2) (5 points) The linear programming relaxation of the original BIP problem gives the optimal solution $(x_1, x_2, x_3) = (1, 0.5, 1)$. Using the branch-and-bound method, we first branch on the variable x_2 . Solve the linear programming relaxation of the subprogram formulated from the branch $x_2 = 0$. Give the optimal solution (x_1^*, x_2^*, x_3^*) and the corresponding optimal value.
- (3) (5 points) Now solve the linear programming relaxation of the subprogram formulated from the branch $x_2 = 1$. Give the optimal solution (x_1^*, x_2^*, x_3^*) and the corresponding optimal value.
- (4) (5 points) What can we conclude for the problem after (2) and (3)? Provide your arguments.