國立臺灣大學115學年度碩士班招生考試試題

無法完全做答的題目，可寫下思路會有部分分數。答題需提供足夠多的細節才會有分數，判斷什麼是足夠多的細節也是考試的一部分。

我們用 $\mathbf{Z}$ = 整數環，$\mathbf{Q}$ = 有理數體，$\mathbf{R}$ = 實數體 以及 $\mathbf{C}$ = 複數體。

Problem 1 (10 pts). Let G be a finite group of order 77.

(1) Show that G contains a normal subgroup of order 7.

(2) Prove that G must be abelian.

Problem 2 (15 pts). Let $\mathbb{F}_3$ be the finite field of three elements. Let

$$\mathrm{SL}_2(\mathbb{F}_3) = \{A \in \mathrm{M}_2(\mathbb{F}_3) \mid \det A = 1\}.$$

(1) Find the cardinality $\#\mathrm{SL}_2(\mathbb{F}_3)$

(2) Find all 3-Sylow subgroups of $\mathrm{SL}_2(\mathbb{F}_3)$.

Problem 3 (20 pts). Let

$$\mathbf{Z}[\sqrt{3}] := \left\{x + y\sqrt{3} \mid x, y \in \mathbf{Z}\right\} \subset \mathbf{R}.$$

be a commutative ring.

(1) For any $a, b \neq 0 \in \mathbf{Z}[\sqrt{3}]$, show that there exists $q \in R$ such that

$$\left|\frac{a}{b} - q\right| \leq \frac{1+\sqrt{3}}{2}.$$

Deduce that

$$a = bq + r, \quad r \in \mathbf{Z}[\sqrt{3}] \text{ and } |\mathrm{N}(r)| < |\mathrm{N}(b)|.$$

Use this division algortithm to use that $\mathbf{Z}[\sqrt{3}]$ is a PID.

(2) Prove that $(\mathbf{Z}[\sqrt{3}])^\times = \pm(2+\sqrt{3})^{\mathbf{Z}}$.

Problem 4 (15 pts). Let F be a field and $R = \mathrm{M}_n(F)$ be the ring of n by n matrices. Prove that the only proper two-sided ideal of R is $\{0\}$.

Problem 5. Let $f(x) = x^3 - 31x - 62 \in \mathbf{Q}[x]$. Let $\alpha_1, \alpha_2, \alpha_3$ be three roots of $f(x)$ in $\mathbf{C}$ and let $E = \mathbf{Q}(\alpha_1, \alpha_2, \alpha_3)$ be the splitting field of $f(x)$ in $\mathbf{C}$. We consider the Galois extension $E/\mathbf{Q}$.

(1) (5 pts) Prove $f(x)$ is an irreducible polynomial in $\mathbf{Q}[x]$.

(2) (5 pts) Put

$$\Delta := (\alpha_1 - \alpha_2)(\alpha_2 - \alpha_3)(\alpha_3 - \alpha_1).$$

Show that $\Delta \in \mathbf{Z}$ is an integer.

(3) (5 pts) Prove that $\#(\mathrm{Gal}(E/\mathbf{Q})) \leq 6$.

(4) (10 pts) Prove $\mathrm{Gal}(E/\mathbf{Q}) \simeq \mathbf{Z}/3\mathbf{Z}$.

Problem 6 (15 pts). Let $f(x) \in \mathbf{Q}[x]$ be an irreducible polynomial of odd degree. Let $\alpha \neq \beta \in \mathbf{C}$ be two distinct roots of $f(x)$. Prove that $\{1, \alpha, \beta\}$ are linearly independent over $\mathbf{Q}$.

試題隨卷繳回