- 請先在試卷第一頁繪製以下表格，然後將答案填入。
- 答案需化至最簡（分數請有理化或化為最簡分數），未以最簡分數表示不予給分。

1 (10%)		6 (10%)	
2 (10%)		7 (10%)	
3 (10%)		8 (10%)	
4 (10%)		9 (10%)	
5 (10%)		10 (10%)	

1. (10%) Consider a 2-dimensional random walk. The starting state is (0,0). In general, if the current state is (i,j), it can move to state $(i+1,j+1)$, $(i+1,j-1)$, $(i-1,j+1)$, $(i-1,j-1)$, or (i,j). What is the number of possible states after n steps? ________

2. (10%) Let there be N functions from A to $\{1,2,3,4,5,6,7,8,9\}$, where A is a set of functions from $\{1,2,3,4,5\}$ to $\{1,2,3,4,5,6\}$. Calculate $N \bmod 11$: ________

3. (10%) Derive the solution for a_n that satisfies the recurrence equation $a_n = 6a_{n-1} - 11\,a_{n-2} + 6\,a_{n-3}$ with $a_0 = 1$, $a_1 = 2$, and $a_1 = 2$: ________

4. (10%) The generating function in *partial fraction decomposition* for the above recurrence equation is ________

5. (10%) Let N be he number of non-negative integer solutions of $x_1 + x_2 + \cdots + x_{12} \leq 6$. Calculate $N \bmod 11$: ________

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6. (10%) The following linear system of three unknowns has a non-zero solution if and only if $\lambda =$________:

$$\begin{cases} x_1 + 2x_2 - 2x_3 = 0, \\ 2x_1 - x_2 + \lambda x_3 = 0, \\ 3x_1 + x_2 + x_3 = 0. \end{cases}$$

7. (10%) Let

$$A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$$

and suppose that

$$A^{2004} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then, $\frac{a+b+c+d}{5^{2004}} =$________.

8. (10%) Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Then, the minimum-norm least-squares solution of the linear system $Ax = b$ is ________.

9. (10%) Let A be an $n \times n$ positive semi-definite matrix with eigenvalues $\lambda_1, \dots, \lambda_n$ and corresponding eigenvectors $v_1, \dots, v_n$. Suppose that $\lambda_1 > \lambda_2 > \dots > \lambda_n$ and that the eigenvectors have unit 2-norms. Let c be a strictly positive real number. Then, the minimum value of the function $f(X) = \mathrm{trace}(AX)$ over all $n \times n$ Hermitian matrices X satisfying $\|X\|_* \le c$ is attained at $X =$________. Here, $\|X\|_*$ denotes the nuclear norm of the matrix X, i.e., the sum of the singular values of X. Express your answer in terms of λ_i, v_i, v_i^*, c, and numbers, where v_i^* denotes the conjugate transpose of v_i.

10. (10%) We say that a function f is L-smooth relative to another function g for some positive real number L over a set $\mathcal{X}$ if the matrix $\nabla^2(Lg - f)(x)$ is positive semidefinite for all $x \in \mathcal{X}$. Now, let

$$f(x) = -\alpha \ln(a_1^T x) - \beta \ln(a_2^T x),$$

where α and β are strictly positive real numbers and a_1 and a_2 are entry-wise strictly positive real n-dimensional vectors. Here, a^T and b^T denote the transposes of a and b. Let

$$g(x) = -\sum_{i=1}^{n} \ln(x[i]),$$

where $x[i]$ denotes the i-th entry of the vector x. Then, the smallest possible L such that the function f is L-smooth relative to g over the set

$$\{v \in \mathbb{R}^d \mid v[i] > 0 \text{ for all } i, v[1] + v[2] + \dots + v[n] = 1\}$$

is given by $L =$________. Express your answer in terms of α, β, a_1, a_2, a_1^T, a_2^T, and numbers.