國立臺灣大學 115 學年度碩士班招生考試試題

※ 注意：請於試卷內之「非選擇題作答區」作答，並應註明作答之題號。

- Any device with a computer algebra system is prohibited during the exam.
- There are FOUR questions in total. Label the question numbers **clearly** on your work.
- Answer all questions. You will have to **show all of your calculations or reasoning** to obtain credits.

1. (30%) Let $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable functions satisfying

$$\begin{aligned} &\text{(I)} \quad f'(x) = 3f(x) + 4g(x) \quad \text{for all } x \in \mathbb{R}, \\ &\text{(II)} \quad g'(x) = -4f(x) + 3g(x) \quad \text{for all } x \in \mathbb{R}, \\ &\text{(III)} \quad f(0) = 0, \qquad g(0) = 2. \end{aligned}$$

(a) Consider

$$H(x) = \big(f(x) - 2e^{3x}\sin(4x)\big)^2 + \big(g(x) - 2e^{3x}\cos(4x)\big)^2$$

Prove that $H'(x) = kH(x)$ for some constant k and find the value of k.

(b) Hence, find $f(x)$ and $g(x)$ explicitly.

2. (30%)

(a) Find the Maclaurin series of $\dfrac{1}{27+x^3}$. Specify the radius of convergence R.

(b) Let $I_m = \displaystyle\int_0^3 \frac{x^m}{27+x^3}\,\mathrm{d}x$ with $m \geq 0$.

(i) For each integer $m \geq 0$, express I_m as

$$I_m = \sum_{k=0}^{\infty} (-1)^k \cdot a_{k,m} \text{ where } a_{k,m} > 0.$$

Find $a_{k,m}$.

(ii) Hence, by using a partial fraction decomposition, evaluate the sum

$$\sum_{k=0}^{\infty} (-1)^k \cdot \frac{108k+45}{(3k+1)(3k+2)}.$$

3. (20%) Evaluate the following integrals.

(a) $\displaystyle\iint_R (x+y)^2\,\mathrm{d}A, \qquad R = \{(x,y) \in \mathbb{R}^2 : \ |x| + |y| \leq 1\}.$

(b) $\displaystyle\iiint_D (x+y+z)^2\,\mathrm{d}V, \qquad D = \{(x,y,z) \in \mathbb{R}^3 : \ |x| + |y| + |z| \leq 1\}.$

4. (20%) Consider a smooth real-valued function $U(x_1, x_2, x_3)$ such that

$$\frac{\partial U}{\partial x_1} > 0, \qquad \frac{\partial U}{\partial x_2} > 0, \qquad \frac{\partial U}{\partial x_3} > 0.$$

for $x_1, x_2, x_3 \geq 0$. We wish to maximize $U(x_1, x_2, x_3)$ subject to the constraints

$$P_1x_1 + P_2x_2 + P_3x_3 \leq I, \qquad x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,$$

where $P_1, P_2, P_3, I > 0$.

(a) Explain, in one or two sentences, why the maximum value of U exists in this setting.

(b) Explain why, at a maximizer (x_1^*, x_2^*, x_3^*) of the above optimization problem, one must have

$$P_1x_1^* + P_2x_2^* + P_3x_3^* = I.$$

(c) Now treat P_1, P_2, P_3, I as parameters. Let $U^*(P_1, P_2, P_3, I)$ be the maximum value of U subject to the constraints. Determine, with explanations, the signs (positive/negative/zero) of

$$\frac{\partial U^*}{\partial P_1}, \quad \frac{\partial U^*}{\partial P_2}, \quad \frac{\partial U^*}{\partial P_3}, \quad \frac{\partial U^*}{\partial I}.$$

In one or two sentences, explain what these results mean in a real-life/commercial context.

試題隨卷繳回