1. (30%) Consider the 2×2 matrix A defined by

$$\mathbf{A}=\begin{pmatrix}\cos(2\alpha) & \sin(2\alpha)\\ \sin(2\alpha) & -\cos(2\alpha)\end{pmatrix} \qquad \alpha\in\mathrm{R}.$$

(a) (5%) Find the determinant of the matrix **A**.

(b) (5%) Find the inverse of the matrix **A**.

(c) (5%) Let $\mathbf{x}=\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$ and suppose that $\mathbf{Ax}=\begin{pmatrix}\cos\beta\\ \sin\beta\end{pmatrix}$. Determine β in terms of α and θ.

(d) (5%) Find all real eigenvalues of the matrix **A**.

(e) (10%) For each eigenvalue found in part (d), find a corresponding real eigenvector.

2. (35%) Let $y=y(x)$ be a function of the variable x.

(a) (10%) Solve the initial value problem $y''+2y'+3y=0;\ y(0)=2, y'(0)=-3$.

(b) (10%) Solve $y''-12y'+36y=0$.

(c) (15%) Find the general solution of $y''+4y=e^x$.

3. (35%) Solve the following partial differential equation along with the boundary conditions.

$$\nabla^2 u=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\left(\frac{\partial^2 u}{\partial \theta^2}\right)=0 \text{ for } 0<r<\frac{5}{3}, -\pi<\theta<\pi.$$

$$\theta \text{ boundary conditions} \begin{cases} u(r,-\pi)-u(r,\pi)=0,\\ \dfrac{\partial u}{\partial \theta}(r,-\pi)-\dfrac{\partial u}{\partial \theta}(r,\pi)=0,\end{cases}$$

$$r \text{ boundary conditions} \begin{cases} u\left(\dfrac{5}{3},\theta\right)=\ln 2+4\cos 3\theta, -\pi<\theta<\pi,\\ \lim\limits_{r\to 0} u(r,\theta)<\infty, -\pi<\theta<\pi.\end{cases}$$

(a) (5%) What is the order of this equation?

(b) (30%) Solve the above equation, using separation of variables.