

考試科目	基礎數學	系所別	統計學系	考試時間	2 月 6 日(五) 第一節
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Part I: Calculus

1. Calculate the following infinite series:

(a) (10%) $\sum_{n=0}^{\infty} (-1)^n \frac{n+1}{3^n}$;

(b) (10%) $\sum_{n=0}^{\infty} \left(\frac{(-1)^n}{n+1} + \frac{(-1)^n}{n+2} \right)$.

2. (10%) Evaluate the integrals $\int_0^{2\pi} \cos^2(x) dx$ and $\int_0^{2\pi} \cos^2(x) dx$.

3. (10%) Let

$$\Gamma(n, k) = \int_0^1 (-\ln x)^{k-1} x^{n-1} dx, \quad n, k \in \mathbb{N}.$$

Express $\Gamma(n, k)$ in terms of the Gamma function $\Gamma(k)$.*Hint:* Recall that $\Gamma(k) = (k-1)\Gamma(k-1)$ and $\Gamma(1) = 1$.4. (10%) Evaluate $\lim_{x \rightarrow 0} \frac{e^x - e^{\sin(x)}}{x - \sin(x)}$.

Part II: Linear Algebra

5. (20%) Determine whether each statement is true or false. No explanation is required.

- (a) If two matrices have the same row space, then they have the same rank.
 (b) If $Ax = 0$ has only the trivial solution, then A is invertible.
 (c) Every spanning set of a vector space must be linearly independent.
 (d) The eigenvalues of a symmetric matrix are always real.

6. Consider A an $n \times n$ matrix.(20%) (a) Suppose that $A^2 = A$. Show that

$$\mathbb{R}^n = \text{Range}(A) \oplus \text{Null}(A),$$

where \oplus denotes the direct sum of subspaces.(b) Suppose further that $A^T = A$. Show that

$$\mathbb{R}^n = \text{Range}(A) \oplus^{\perp} \text{Null}(A),$$

where \oplus^{\perp} denotes the orthogonal direct sum.

7. (10%) Let

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

Determine if A is positive definite.

備

註

- 一、作答於試題上者，不予計分。
 二、試題請隨卷繳交。