考 試 科 目	微積分	系 所 別	應用數學系	考 試 時 間	2 月 6 日(五) 第 三 節

Show all your work to earn the credits.

1. (8 points) Evaluate the integral: $\int \frac{\ln(1+x^2)}{x^2}dx$.

2. (8 points) Evaluate the integral: $\int_0^{3\sqrt{3}} \frac{x^3}{\sqrt{x^2+9}}dx$.

3. (8 points) Evaluate the double integral: $\iint_R x^2 dA$, where R is the region enclosed by the ellipse $\frac{x^2}{4} + y^2 = 1$ and above the x-axis.

4. (8 points) Let $\vec{F}(x,y,z) = < xy^2, yz^2, zx^2 >$ and let S be the union of the upper hemisphere of radius 2 centered at the origin with the disc of radius 2 in the xy-plane centered at the origin such that S is positively oriented. Evaluate the surface integral: $\iint_S \vec{F}(x,y,z)\cdot d\vec{S}$.

5. Let $\vec{F}(x,y,z) = < y, x+e^z, ye^z + 1 >$.

   (a) (5 points) Prove that $\vec{F}$ is conservative.

   (b) (5 points) Find a potential function for $\vec{F}$.

   (c) (8 points) Evaluate the line integral: $\int_C \vec{F}\cdot d\vec{r}$, where C is given by $\vec{r}(t) = < t^2, t^3, \sin(\pi t) >$ for $0 \le t \le 2$.

6. (10 points) Find the maximum and minimum values of the function $f(x,y) = xy$ subject to the constraint $3x^2 + y^2 = 6$.

7. (10 points) Find the following limit if it exists, or show that the limit does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^6}{(x^2+y^2)^4}.$$

備 註	一、作答於試題上者，不予計分。 二、試題請隨卷繳交。

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8. (10 points) Determine the smallest positive integer p for which the series

$$\sum_{n=1}^{\infty} \frac{(1+\frac{1}{pn})^{n^2}}{(\arctan n)^n}$$

converges. Provide a justification for your answer.

9. A function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is said to be Lipschitz continuous on $A \subseteq \mathbb{R}$, if there exists a real number $K \geq 0$ such that $|f(x) - f(y)| \leq K|x - y|$ for any $x, y \in A$.

   (a) (5 points) Prove that the function $g(x) = \dfrac{x+1}{x+3}$ is Lipschitz continuous on $[0, \infty)$.

   (b) (5 points) Use the $\epsilon - \delta$ definition to show that if f is Lipschitz continuous on $[a, b]$, then f is continuous on $[a, b]$.

10. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence with $a_1 = 4$ and

$$a_{n+1} = \frac{2}{1+a_n} \quad \text{for all } n \geq 1.$$

   (a) (5 points) Show that $\{a_{2n}\}_{n=1}^{\infty}$ is an increasing and bounded sequence.

   (b) (5 points) Show that $\{a_n\}_{n\geq 1}$ is a convergent sequence.

備 註	一、作答於試題上者，不予計分。 二、試題請隨卷繳交。