國立臺灣大學 114 學年度碩士班招生考試試題

※ 注意：請於試卷內之「非選擇題作答區」依序作答，並應註明作答之大題及小題題號。

1. (5%) Let A be an arbitrary $n \times n$ matrix over the field of real numbers $\mathcal{R}$, and let 't' denotes the transpose. Answer the following questions with "True" or "False".
   (a) If A is diagonalizable, then A has n distinct eigenvalues.
   (b) If A has n distinct eigenvalues, then A is diagonalizable.
   (c) The nullity of A equals the nullity of A^t.
   (d) $\det A = \det\sqrt{A^tA}$, where 'det' denotes the determinant.
   (e) If there exists an $n \times n$ matrix B satisfying $AB = I_n$ (where I_n denotes the $n \times n$ identity matrix), then A is invertible.
   (Getting 5 points if all answers are correct. Otherwise, 0 point.)

2. (5%) Answer the following questions with "True" or "False".
   (a) A nonzero subspace of $\mathcal{R}^n$ has a generating set.
   (b) A nonzero subspace of $\mathcal{R}^n$ has a basis.
   (c) A nonzero subspace of $\mathcal{R}^n$ has a unique generating set.
   (d) A nonzero subspace of $\mathcal{R}^n$ has a unique basis.
   (e) Suppose $\mathcal{S}$ is a linearly independent set with size being equal the dimension of a nonzero subspace $\mathcal{V}$. Then, $\mathcal{S}$ is a basis for $\mathcal{V}$.
   (Getting 5 points if all answers are correct. Otherwise, 0 point.)

3. (5%) Answer the following questions with "True" or "False".
   (a) Let U be a subset of a subspace $\mathcal{R}^n$. Let $U^\perp$ be the orthogonal complement of U. Then, $(U^\perp)^\perp = U$.
   (b) $U^\perp$ is a subspace of $\mathcal{R}^n$.
   (c) The intersection of a U and its orthogonal complement $U^\perp$ is an empty set.
   (d) Let $T : \mathcal{R}^n \to \mathcal{R}^m$ be a linear transformation. Then, the null space of T equals the $(\text{range}\, T)^\perp$, where range denotes the range space.
   (e) If two vectors are orthogonal, they are linearly independent.
   (Getting 5 points if all answers are correct. Otherwise, 0 point.)

4. (5%) Let A be an $n \times n$ matrix over the field of real numbers $\mathcal{R}$ satisfying $A^2 + A = 0$. Determine all possible values of the determinant of A for any n, respectively.

5. (5%) Let

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

   Find a general form of A^k for any integer k.

6. (15%) Consider a 2×2 matrix over the field of real numbers $\mathcal{R}$.

$$A = \begin{bmatrix} a & 1-b \\ 1-a & b \end{bmatrix}.$$

   (a) (5%) Determine the eigenvalues of A.
   (b) (5%) Determine a basis for each eigenspace of A.
   (c) (5%) Under what conditions is A diagonalizable?

7. (10%) Let A be an $n \times n$ matrix of the following form:

$$\begin{bmatrix} n+2 & n+2 & n+2 & \cdots & n+2 \\ 1 & 3 & 1 & \cdots & 1 \\ 1 & 1 & 3 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 3 \end{bmatrix}.$$

   Calculate $\det A$.

見背面

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8. (25%) A gacha game announces a new character, Monty Hall, who can be summoned by collecting 100 Monty Hall coins. To obtain Monty Hall coins, players buy lottery tickets for 10 emeralds each. (Emerald is the in-game currency.) Each ticket has a 50% chance of winning one Monty Hall coin, and a 50% chance of winning nothing.

(a) (3%) Compute the average cost, in emeralds, of summoning Monty Hall.

(b) (4%) To avoid players quitting the game due to bad luck, the game developers hold a meeting. They plan to give out one Monty Hall coin for every three losing tickets (a losing ticket is a lottery ticket that does not win any Monty Hall coin). Compute the new average cost, in emeralds, of each Monty Hall coin.

(c) (5%) Estimate the new average cost, in emeralds, of summoning Monty Hall. Round your answer to the nearest integer.
(Hint: A player might have $3k+1$ or $3k+2$ losing ticket when Monty Hall is summoned. The extra one or two losing tickets will be deleted by the system.)

(d) (6%) One of the game developers, Simpson, believes that the new plan is too generous because it gives out Monty Hall coins even when players are not frustrated. For example, a player who buys five tickets may get the following outcomes: **lose, win, lose, win, lose**. Now, because the fifth ticket is the third losing ticket, the player would be given a coin. But the player may not feel frustrated because the fourth ticket is a win.
Simpson proposes to give out one Monty Hall coin for every three **consecutive** losing tickets. That is, **lose, win, lose, win, lose** will not be given extra coins. But **lose, wins, lose, lose, lose** will be given one extra coin after the fifth ticket because it is the third losing ticket in a row. Note that, once the player is given a coin, the count of losing tickets resets. For example, **lose, lose, lose, lose, lose, lose** will be given two extra coins, one after the third ticket and the other after the sixth ticket.
Estimate the average cost, in emeralds, of each Monty Hall coin assuming Simpson's plan.

(e) (7%) Due to miscommunication, the programmer, Bertrand, implements Simpson's proposal incorrectly. Instead of resetting the counter when a coin is given, Bertrand's implementation gives out four Monty Hall coins for **lose, lose, lose, lose, lose, lose**, one after the third ticket, one after the fourth ticket, one after the fifth ticket, and one after the sixth ticket.
Estimate the average cost, in emeralds, of summoning Monty Hall assuming Bertrand's incorrect implementation that does not reset the counter. Round your answer to the nearest integer.

9. (25%) Albert, Issac, and Thomas are the top three chip manufacturers in the world. Each of their chips is made of billions of transistors. Each transistor has an independent probability of being bad. The probability is denoted by p_A, p_I, and p_T for Albert, Issac, and Thomas, respectively. A chip is considered good if it contains no bad transistors. The **yield** of a chip is the probability that a chip is good.

(a) (3%) Issac is producing a chip with 8 billion ($8 \cdot 10^9$) transistors with $p_I = 2.5 \cdot 10^{-11}$. Estimate the yield of Issac's chip to three significant digits. ($9 \cdot 10^{-1}$ has one significant digit, $9.9 \cdot 10^{-1}$ has two significant digits, and $9.99 \cdot 10^{-1}$ has three significant digits. Euler's number is $e = 1/0! + 1/1! + 1/2! + 1/3! + ... = 2.71828$.)

(b) (4%) Thomas is producing a chip with 5 billion transistors and a yield of 90%. Estimate p_T to three significant digits. ($\ln(2) = 0.69315$, $\ln(3) = 1.09861$, $\ln(5) = 1.60944$.)

(c) (5%) Albert announces that its three-core chip has a yield of 31.25%. Each core has 25 billion transistors. Estimate p_A to three significant digits.

(d) (6%) It is later revealed that Albert designs the chip to have four cores but only use three. Here is a breakdown of the manufacturing process:
- If all four cores are good, the chip is considered good. One core is disabled and the chip is sold as a three-core chip.
- If three cores are good and the other core contains bad transistors, the chip is still considered good. The bad core is disconnected and the chip is sold as a three-core chip.
- If two or fewer cores are good, the chip is considered bad and discarded.

Given the new information that 31.25% is the probability that a four-core chip has three or more good cores, estimate p_A to three significant digits.

(e) (7%) There are many ways Albert can produce chips with three good cores:
- Design a three-core chip and hope that all three cores are good.
- Design a four-core chip and disable/disconnect one core.
- Design a five-core chip and disable/disconnect two cores.
- Design a six-core chip and disable/disconnect three cores.
- ...

Suppose that the only cost in chip manufacturing is the cost of transistors. Which option is the most cost-effective for Albert to produce chips with three good cores? That is, which produces the largest amount of three-good-core chips per transistor cost? Assume the p_A you estimated in part **(d)**.