國立臺灣師範大學 114 學年度碩士班招生考試試題

科目：高等微積分　　　　　　　　　　　　適用系所：數學系

注意：1.本試題共 1 頁，請依序在答案卷上作答，並標明題號，不必抄題。2.答案必須寫在指定作答區內，否則依規定扣分。

1. (10 分) Let E be a nonempty bounded subset of $\mathbb{R}$ and $\alpha = \sup E$. Show that there is an increasing sequence $x_1, x_2, \ldots, x_n, \ldots$ in E such that $\lim_{n\to\infty} x_n = \alpha$.

2. (10 分) Determine the limit:

$$\lim_{x\to\infty}\left(\sqrt{x^2+x+1}-\sqrt{x^2-x+1}\right).$$

3. (10 分) Let C be a compact set in a metric space (M,d). Show that C is bounded and closed.

4. (10 分) Consider the function $f(x) = \dfrac{x^2}{1+x^2}$ for $x \in \mathbb{R}$. Is f uniformly continuous on $\mathbb{R}$? Justify your answer.

5. (10 分) Let f, g be two differentiable real functions on $\mathbb{R}$. Suppose that $f'(x) = g(x)$ and $g'(x) = -f(x)$ for all $x \in \mathbb{R}$. Show that $h(x) = (f(x))^2 + (g(x))^2$ is a constant function on $\mathbb{R}$.

6. (10 分) Evaluate the infinite series: $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k(k+2)(k+3)}$.

7. (10 分)Suppose that f is a bounded real function on $[0,1]$ such that $g(x) = (f(x))^3$ is Riemann integrable on $[0,1]$. Show that $f(x)$ is also Riemann integrable on $[0,1]$.

8. (10 分) Find the radius R of convergence for the power series $\displaystyle\sum_{n=1}^{\infty} C_n^{2n} x^n$. Does the series converge at $x = R$? (Remark. $C_n^{2n} = \dfrac{(2n)!}{(n!)^2}$ is the binomial coefficient.)

9. (10 分) For any continuous function f on $[0,1]$, define

$$F(y) = \int_0^1 f(x)\cos(xy)\,\mathrm{d}x, \qquad 2 < y < 4.$$

Show that F is differentiable in the open interval $(2,4)$.

10. (10 分) Let $B = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leqslant 1\}$ be the solid unit ball in $\mathbb{R}^3$. Evaluate the triple integral:

$$\iiint_B x^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

(試題結束)