國立臺灣師範大學 114 學年度碩士班招生考試試題

科目：基礎數學　　　　　　　　　　　　　適用系所：數學系

注意：1.本試題共 2 頁，請依序在答案卷上作答，並標明題號，不必抄題。2.答案必須寫在指定作答區內，否則依規定扣分。

Part I: Calculus（無計算過程或說明不給分，總分 50 分）

1. Do the following two questions.

   (a) (6 pts) Explain why the improper integral $\int_2^{\infty} (\frac{x+2}{x-1})^x \, dx$ is divergent.

   (b) (6 pts) Evaluate the limit $\lim_{x\to\infty} \frac{\int_2^x (\frac{t+2}{t-1})^t \, dt}{x}$.

2. Let $f(x) = (\ln x)(\ln(\ln x)) - 1$.

   (a) (4 pts) Find the domain of f.

   (b) (4 pts) Find the open interval where the graph of f is increasing.

   (c) (5 pts) Show that f has exactly one zero in its domain.

3. (9 pts) Find the maximum value of the function $f(x, y, z) = x + 2y + 3z$ on the curve of intersection of the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$.

4. (8 pts) Compute the integral $\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2 \, dydx$.

5. (8 pts) Let (r, θ) be the polar coordinate. Find the area of the region bounded by the two curves: $r = 1 - \cos\theta$ and $r = 1 + \cos\theta$.

(尚有試題)

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Part II. Linear Algebra （總分 50 分）

1. (24 pts) Let $\{\mathbf{b}_1, \ldots, \mathbf{b}_5\}$ and $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis for $\mathbb{R}^5$ and $\mathbb{R}^3$ respectively. Let $T : \mathbb{R}^5 \to \mathbb{R}^3$ be a linear transformation satisfying

$$T(\mathbf{b}_1) = \mathbf{e}_1 + 6\,\mathbf{e}_3, \quad T(\mathbf{b}_2) = -\mathbf{e}_1 + 2\,\mathbf{e}_2 \ \text{ and } \ T(\mathbf{b}_4) = 3\,\mathbf{e}_1 - 4\,\mathbf{e}_2 + 5\,\mathbf{e}_3.$$

Suppose that T is defined by $T(\mathbf{x}) = A\mathbf{x}$ for some 3×5 matrix A and that the reduced row echelon form of A is given as follows:

$$R = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}.$$

(a) Find a basis for the null space of T.

(b) Find the matrix A.

(c) Show that there exists a linear transformation $S : \mathbb{R}^3 \to \mathbb{R}^5$ such that $T \circ S : \mathbb{R}^3 \to \mathbb{R}^3$ is the identity transformation on $\mathbb{R}^3$. Is S unique? You need to explain or verify your answer.

2. (12 pts) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation of rank r and let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for $\mathbb{R}^n$. Define $\mathcal{N} = \{(a_1, \ldots, a_n) \mid a_1 T(\mathbf{v}_1) + a_2 T(\mathbf{v}_2) + \cdots + a_n T(\mathbf{v}_n) = \mathbf{0}_m\}$ where $\mathbf{0}_m$ denotes the zero vector of $\mathbb{R}^m$.

(a) Show that $\mathcal{N}$ is a subspace of $\mathbb{R}^n$ and that $\dim \mathcal{N} = n - r$.

(b) Let R be the range of T and let I be any linear independent subset of $\mathbb{R}^m$. Show that $|I \cap R| \le r$ where $|A|$ denotes the number of elements in the set A.

3. (14 pts) Let $A = [a_{ij}]$ be an $n \times n$ matrix where $a_{ij} = \begin{cases} 5, & \text{if } i = j; \\ 3, & \text{if } i - j = 1; \\ 2, & \text{if } j - i = 1; \\ 0, & \text{otherwise.} \end{cases}$

That is $A = \begin{bmatrix} 5 & 2 & 0 & \cdots & 0 \\ 3 & 5 & 2 & & \vdots \\ 0 & 3 & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 2 \\ 0 & \cdots & 0 & 3 & 5 \end{bmatrix}$.

(a) Find the determinant of A.

(b) For the case where $n = 3$, determine whether or not A is diagonalizable. You need to explain your answer.

(試題結束)