

考 試 科 目	基礎數學	系 所 別	統計學系	考 試 時 間	2 月 12 日 (三) 第 一 節
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Part I: Calculus

- (15%) Use the ϵ - δ argument to show that $f(x) = \sqrt[3]{x}$, $x \in \mathbb{R}$, is continuous at any arbitrary point $c \neq 0$. Hint: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.
- (10%) Evaluate the following limits:
 - $\lim_{n \rightarrow \infty} \frac{1}{n^3} \left(\sum_{i=1}^{3n} \sqrt{i} \right)^2$;
 - $\lim_{x \rightarrow 1} \frac{\log x - x^x + 1}{\log x - x + 1}$.
- (15%) Let $\{a_n\}_{n \geq 0}$ be a sequence of real numbers such that $a_0 > 0$ and $a_{n+1} = \frac{1 + \sqrt{1 + a_n^2}}{a_n}$. Let $\{b_n\}_{n \geq 0}$ with $b_0 \in \left(0, \frac{\pi}{2}\right)$ be a sequence satisfying $a_n = \tan(b_n)$. Find b_n in terms of b_0 .
- (10%) Evaluate the integral: $\int_0^\infty e^{-\frac{z^2}{2}} dz$. Hint: $\left(\int_0^\infty e^{-\frac{z^2}{2}} dz \right)^2 = \int_0^\infty \int_0^\infty e^{-\frac{(x^2+y^2)}{2}} dx dy$.

Part II: Calculus

- (20%) For each of the following statements, determine whether it is true (O) or false (X). Do not give explanation.
 - If all the equations in one system of linear equations are linear combinations of the equations in another system of linear equations, then the solution sets of the two systems are necessarily identical.
 - A product of invertible matrices must be invertible.
 - The zero vector is always included in the basis of a vector space.
 - All rows of a invertible row-reduced echelon matrix form a basis for the row space of the matrix.
- (10%) Let A be an $n \times n$ matrix with entries $A_{i,j}$ all real values. Prove that $A = 0$ if and only if the trace value $\text{tr}(A^T A) = 0$.
- (10%) If A is an $n \times n$ matrix, show that if $\text{Null}(A^T) = \text{Range}(A - I)$, then $A = A^T$.
- (10%) If A is an $n \times n$ matrix with $A^2 = A$ and $A^T = A$. Prove that
 - if $x \in \text{Null}(A^T)$, then $x \in \text{Range}(A - I)$;
 - if $x \in \text{Range}(A - I)$, then $x \in \text{Null}(A^T)$.

備

註

- 一、作答於試題上者，不予計分。
- 二、試題請隨卷繳交。