國立交通大學100學年度碩士班考試入學試題

科目：微分方程與線性代數(1212)　　考試日期：100年2月17日 第 2 節

系所班別：電機工程學系　　組別：電機系　　

【可使用計算機】＊作答前請先核對試題、答案卷(試卷)與准考證之所組別與考科是否相符！！

1. (25%) Let $x(t)$ be a function of t satisfying the following differential equation

$$tx''(t) - 2x'(t) + tx(t) = 0.$$

(a) (15%) Assuming $x(0) = 1$, find the Laplace transform $X(s)$ of $x(t)$.

(b) (10%) Continued from part **(a)**, find $x(t)$ through the inverse Laplace transform of $X(s)$.

2. (10%) Find the general solution of the following nonhomogeneous linear differential equation

$$x''(t) + x(t) = 2\sec(t).$$

3. (10%) Let $x_1(t)$ and $x_2(t)$ be functions of t satisfying the following system of differential equations

$$\begin{cases} x_1''(t) + x_2''(t) + 2x_1'(t) + x_2'(t) + x_1(t) &= 0 \\ x_1''(t) + x_2''(t) + 2x_2'(t) + x_2(t) &= 0 \end{cases}$$

Assuming $x_1(0) = x_2(0) = 1$ and $x_1'(0) = x_2'(0) = 0$, find the solutions $x_1(t)$ and $x_2(t)$.

4. (5%) Solve the following initial value problem

$$tx'(t) = x(t) + \sqrt{t^2 - x^2(t)}, \quad x(t_0) = 0$$

for some $t_0 > 0$.

5. (16%) Consider the linear equations below:

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{1}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \tag{2}$$

$$\begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \tag{3}$$

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \tag{4}$$

(a) (4%) Which equations are consistent? Explain. (A linear equation is consistent if it has a solution.)

(b) (4%) For each equation that is consistent, find a solution that has the least (Euclidean) norm.

(c) (4%) Among the equations that are not consistent, determine the ones that have a unique least squares solution? Explain.

(d) (4%) For each equation you found in **(c)**, compute the least squares solution.

6. (9%) Consider the function $f(x_1, x_2, x_3) = x_1^2 - x_3^2 - 4x_1x_2 + 4x_2x_3$, where x_1, x_2, x_3 are real numbers.

(a) (3%) Express the function as a quadratic form, that is, find a symmetric matrix A such that $f(x_1, x_2, x_3) = x^T Ax$, where $x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$.

(b) (6%) Let

$$f_{\min} = \min_{x_1,x_2,x_3} f(x_1, x_2, x_3) \text{ subject to } x_1^2 + x_2^2 + x_3^2 = 1$$

and

$$f_{\max} = \max_{x_1,x_2,x_3} f(x_1, x_2, x_3) \text{ subject to } x_1^2 + x_2^2 + x_3^2 = 1.$$

Find $f_{\min}$ and $f_{\max}$.

7. (11%) Suppose A is a 3×3 real matrix, and satisfies

$$(A^2 + A + I)(A + 2I) = 0 \tag{5}$$

where I is the 3×3 identity matrix.

(a) (3%) Show that A is nonsingular.

(b) (3%) Express A^{-1} as a polynomial of A. Is your expression unique?

(c) (2%) Let **V** be the set of all 3×3 real matrices satisfying (5). Is **V** a vector space? Explain.

(d) (3%) Assume that the eigenvalues of A are distinct, find $\det(A^{-1})$, the determinant of A^{-1}.

8. (8%)

(a) (4%) The 4 vectors

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, v_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

form the so-called *Haar basis* of $\mathbf{R}^4$. Express the vector $x = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}^T$ as a linear combination of the basis vectors.

(b) (4%) Find an eigenvector of the matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

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9. (6%) Are the following statements True (T) or False (F)? For each of your answers give a brief explanation. A correct answer with no justification will receive no credits.

(a) (2%) If A and B are similar matrices, then $\det(A) = \det(B)$.

(b) (2%) If two matrices A and B have the same null space, then they must be either both singular or both nonsingular.

(c) (2%) If $y \neq 0$ is such that $A^T y = 0$, then $Ax = y$ has no solution.