國立臺灣師範大學 112 學年度碩士班招生考試試題

科目：基礎數學　　　　　　　　　　　　　　適用系所：數學系

注意：1.本試題共 2 頁，請依序在答案卷上作答，並標明題號，不必抄題。2.答案必須寫在指定作答區內，否則依規定扣分。

Part I: Calculus

1. Evaluate the following limits.

   (a) (4 points) $\lim_{x\to 0} \dfrac{\tan 2x + \sin x}{\sqrt{1+4x} - \sqrt{1-2x}}$.　　(b) (4 points) $\lim_{x\to -\infty} \dfrac{\sqrt{2x^2-4x}}{x}$.

2. (6 points) Let Γ be the plane curve $x^3 + y^3 - 6xy = 0$. Find the slope of the tangent line to Γ at the point $\left(\frac{4}{3}, \frac{8}{3}\right)$.

3. (8 points) Let n be a positive integer, and $P_n(x)$ be the following polynomial:

$$P_n(x) := \frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n.$$

   Show that the equation $P_n(x) = 0$ has n distinct real roots.

4. Consider the function $f(x,y) = 2xy - \frac{1}{2}(x^4+y^4) + 1$.

   (a) (4 points) Find all critical points for $f(x,y)$.

   (b) (4 points) Classify all critical points above as local maxima, local minima, or saddle points.

5. Evaluate the following indefinite integrals.

   (a) (4 points) $\int \sin(\ln x^2)\,\mathrm{d}x$.　　(b) (4 points) $\int \dfrac{1}{2x^2-x+3}\,\mathrm{d}x$.

6. (6 points) Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos\theta$.

7. (6 points) Find the values of α such that the following integral is convergent:

$$\int_0^\infty \frac{\sin x}{x^\alpha}\,\mathrm{d}x.$$

(背面尚有試題)

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Part II : Linear Algebra

1. (20 points) Let $A = \begin{pmatrix} 1 & 2 & 1 & -1 \\ 2 & 5 & 1 & -1 \\ 1 & 3 & 1 & 2 \\ 2 & 4 & 2 & -1 \end{pmatrix}$ be a 4×4 matrix. We know that $\mathbf{v} = \begin{pmatrix} 2 \\ 0 \\ 2 \\ 3 \end{pmatrix}$ is in the column space of A.
   (a) Find the determinant $\det(A)$.
   (b) Find an orthonormal basis of the column space $\mathrm{Col}(A)$ and the null space $\mathrm{Null}(A)$.
   (c) Find the inverse A^{-1}.
   (d) Express $\mathbf{v}$ as a linear combination of the column vectors of A.

2. (20 points) Let $\mathfrak{M}_{2\times 2}$ be the vector space of all 2×2 real matrices and let $\mathsf{P}_2(\mathbb{R})$ be the vector space of polynomials of degree less than or equal to 2. Let β and γ be the ordered bases for $\mathfrak{M}_{2\times 2}$ and $\mathsf{P}_2(\mathbb{R})$, respectively, where

$$\beta = \{\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\}, \quad \gamma = \{x^2, x^2 + x, x^2 + x + 1\}.$$

   Consider the following two functions T_1, T_2 from $\mathfrak{M}_{2\times 2}$ to $\mathsf{P}_2(\mathbb{R})$.

   - T_1 is defined by $T_1(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = bx^2 + 2dx + a + b$.
   - T_2 is the linear transformation whose matrix representation in the ordered bases β and γ is

$$\begin{pmatrix} 3 & 2 & 3 & -2 \\ 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & -1 \end{pmatrix}.$$

   (a) Prove that T_1 is a linear transformation.
   (b) Compute the matrix that represents T_1 in the ordered bases β and γ.
   (c) Find $T_2(\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix})$.
   (d) Find the null space of T_2.

3. (10 points) Let $A \in \mathfrak{M}_{n\times n}$. For any eigenvalue $\lambda \in \mathbb{R}$ of A, let $\mathsf{E}_\lambda(A)$ denote the corresponding eigenspace for A.
   (a) Suppose that λ, λ' are two distinct eigenvalues of A. Prove $\mathsf{E}_\lambda(A) \subseteq \mathsf{E}_{\lambda'}(A^{\mathrm{t}})^{\perp}$.
   (b) Prove that if $\mathbf{v} \in \mathbb{R}^n$ is both an eigenvector of A and an eigenvector of A^{t}, then $A\mathbf{v} = A^{\mathrm{t}}\mathbf{v}$.

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