國立成功大學
111學年度碩士班招生考試試題

編　號：235

系　所：統計學系

科　目：數理統計

日　期：0220

節　次：第 2 節

備　註：不可使用計算機

※ 考生請注意：本試題不可使用計算機。 請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

1. Suppose that a random variable X has the probability density function (pdf) given by

$$f(x;\,\beta)=\begin{cases}1, & \beta=\dfrac{1}{2};\\ \dfrac{2(1-\beta)\tanh^{-1}(1-2\beta)}{1-2\beta}\left(\dfrac{\beta}{1-\beta}\right)^{x}, & \beta\neq\dfrac{1}{2},\end{cases}$$

where $x\in(0,1)$ and $\beta\in(0,1)$ is the shape parameter.

i. (10%) Find the cumulative distribution function (cdf) $F(x,\beta)$ of the random variable X.
ii. (10%) Find the hazard rate function $h(x,\beta)$ of the random variable X.
iii. (10%) Find $\lim_{x\to 1} h(x,\beta)$ and $\lim_{x\to 0} h(x,\beta)$.
iv. (10%) Find the moment generating function (MGF) $M_X(t)$ of the random variable X.
v. (10%) We define the quantile function as $F^{-1}(p,\beta)=Q_p(\beta)$, $p\in(0,1)$, of the distribution. Find $Q_p(\beta)$.
vi. (10%) Find the score function and the Fisher information of β, respectively. Suppose $\hat{\beta}$ is the MLE of β, give the asymptotic $100(1-\theta)\%$ confidence intervals of the β parameter, where $\theta\in(0,1)$.

2. (10%) Let a random variable X have a distribution $f(x;a,b)$ with parameters $b>0$ and $c\in R$, denoted by $X\sim PG(b,c)$, given by

$$f(x;a,b)=\frac{1}{2\pi^2}\sum_{k=1}^{\infty}\frac{g_k}{(k-1/2)^2+c^2/(4\pi^2)},$$

where g_k are independent gamma random variables with the distribution $h(y;b)$ given by

$$h(y;b)=\tfrac{1}{\Gamma(b)}y^{b-1}e^{-y},y>0\,.$$

Show

$$2^{-b}e^{\kappa\psi}\int_0^{\infty}e^{-\frac{\omega\psi^2}{2}}p(\omega)d\omega\;=\frac{\left(e^{\psi}\right)^{a}}{(1+e^{\psi})^{b}},$$

where $\kappa=a-b/2$ and $\omega\sim PG(b,0)$.

3. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the uniform distribution given by

$$f(x;\theta) = \begin{cases} 1, & \theta - \frac{1}{2} < x < \theta + \frac{1}{2} \\ 0, & \text{elsewhere.} \end{cases}$$

Let Y_1 and Y_n be the first and nth order statistic.

i. (10%) Find the correlation of Y_1 and Y_n.

ii. (10%) Show that any estimator $\hat{\theta}$ such that $Y_n - \frac{1}{2} \leq \hat{\theta} \leq Y_1 + \frac{1}{2}$ can serve as a maximum likelihood estimator of θ.

iii. (10%) Show that the sample midrange $\frac{Y_1+Y_n}{2}$ is unbiased and more efficient estimator than the sample mean $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ for θ.