考試科目	基礎數學	系所別	統計所	考試時間	2 月 5 日(五) 第一節

Part I: multiple choice questions (4 points each)

1. Which of the following statements are true?
   (a) f is differentiable at x_0 implies that f is continuous at x_0.
   (b) $f(x) = x^4 + 3$ has an inflection point.
   (c) $\lim_{x\to\infty}(1 + 1/x)^x = 1$.
   (d) $\lim_{x\to\infty} x^{-1/2} \ln x = 0$.
   (e) None of the above.

2. Let $f(x) = \ln(1 + x)$. Which of the following statements are true?
   (a) The Taylor series about $x = 0$ is $x - x^2 + x^3 - x^4 + \cdots$, for $|x| < 1$.
   (b) The Taylor series about $x = 0$ is $x - x^2/2 + x^3/3 - x^4/4 + \cdots$, for $|x| < 1$.
   (c) $f'(x)$ is increasing in x for $x > -1$.
   (d) $f''(x)$ is increasing in x for $x > -1$.
   (e) None of the above.

3. Consider $f(x) = 1/(1 + x)$. Which of the following statements are true?
   (a) The Taylor series about $x = 0$ is $1 + x + x^2 + x^3 + \cdots$, for $|x| < 1$.
   (b) The Taylor series about $x = 0$ is $1 - x + x^2 - x^3 + \cdots$, for $|x| < 1$.
   (c) $\int_{-1}^{1} f(x)dx = \infty$.
   (d) $\int_{-1}^{\infty} f(x)dx = \infty$.
   (e) None of the above.

4. Which of the following statements are true?
   (a) $1/2 + 1/3 + \cdots + 1/n < \ln n$
   (b) $1 + 1/2 + 1/3 + \cdots + 1/n > \ln n$
   (c) $1 + 1/2 + 1/3 + 1/4 + \cdots$ does not converge.
   (d) $1 + 1/2^{1.5} + 1/3^{1.5} + 1/4^{1.5} + \cdots$ does not converge.
   (e) $1/2 - 1/3 + 1/4 - 1/5 + \cdots + (-1)^n/n + \cdots$ does not converge.

5. Let $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$ and $x = (x_1, x_2, x_3)^T$, a 3×1 column vector. Which of the following statements are true?
   (a) The gradient of f with respect to x is $(2x_1, 2x_2, 2x_3)^T$.
   (b) The Hessian of f with respect to x is $(2, 2, 2)^T$.
   (c) The minimum value of f subject to $2x_1 + 2x_2 - x_3 = 7$ is 5.
   (d) The maximum value of f subject to $2x_1 + 2x_2 - x_3 = 7$ is 10.
   (e) None of the above.

6. Let $f(x) = (\sin x)/x$. Which of the following statements are true?
   (a) $\lim_{x\to 0} f(x) = 0$.
   (b) $\lim_{x\to 0} f(x) = 1$.
   (c) $\lim_{x\to \frac{\pi}{2}} f(x) = 0$.
   (d) $\max_x f(x) = 1$.
   (e) None of the above.

備註	一、作答於試題上者，不予計分。 二、試題請隨卷繳交。

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7. Let $F(x) = \int_0^x [(\sin t)/t]dt$. Which of the following statements are true?
   (a) $\lim_{x\to 0} F(x) = 0$.
   (b) $\lim_{x\to 0} F(x) = 1$.
   (c) $\lim_{x\to\infty} F(x) < \pi$.
   (d) $\lim_{x\to\infty} F(x) > \pi$.
   (e) None of the above.

8. Let A be an $m \times n$ matrix. Which of the following statements are true?
   (a) nullity(A) + rank(A)=n.
   (b) rank(A)=rank(A^T).
   (c) If $m = 7$ and $n = 5$, then rank(A) is at most 5.
   (d) Suppose that $m = n$. Then, "A is singular" $\Leftrightarrow$ "A has rank n".
   (e) None of the above.

9. Let A be an $n \times n$ matrix whose (i, j) component is a_{ij}. The trace of A is defined as $\mathrm{tr}A = \sum_{i=1}^n a_{ii}$. Let B and C be $n \times n$ matrices. Which of the following statements are true?
   (a) $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.
   (b) $\mathrm{tr}(ABC) = \mathrm{tr}(CBA)$.
   (c) $\mathrm{tr}(A^T B) = \mathrm{tr}(AB^T)$.
   (d) $\mathrm{tr}(A + B) = (\mathrm{tr}B)(\mathrm{tr}A)$.
   (e) None of the above.

10. Let A be an $n \times n$ matrix whose (i, j) component is a_{ij}. Let f be a real-valued function defined on A. Let $\nabla_A f(A)$ be the gradient of $f(A)$ with respect to A: $\nabla_A f(A)$ is defined as an $n \times n$ matrix whose (i, j) entry is $\partial f/\partial a_{ij}$. Let B and C be $n \times n$ matrices. Which of the following statements are true?
    (a) If $f(A) = \mathrm{tr}(AB)$, then $\nabla_A f(A) = B$.
    (b) If $f(A) = \mathrm{tr}(AB)$, then $\nabla_A f(A) = B^T$.
    (c) If $f(A) = \mathrm{tr}(AA^T C)$, then $\nabla_A f(A) = CA + C^T A^T$.
    (d) If $f(A) = \mathrm{tr}(AA^T C)$, then $\nabla_A f(A) = CA + CA^T$.
    (e) None of the above.

11. Which of the following statements are true?
    (a) $A_{n\times n}$ is singular if and only if $Ax = b$ has infinitely many solutions for every $n \times 1$ vector b.
    (b) Every matrix transformation is a linear transformation.
    (c) If A is a nonsingular upper triangular matrix, then the adjoint matrix is lower triangular.
    (d) The dimension of the zero space $\{\mathbf{0}\}$ is 0.
    (e) None of the above.

12. Let A be a 2×2 matrix defined as $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Which of the following statements are true?
    (a) The characteristic function is $f_A(\lambda) = \lambda^2 - 1$.
    (b) The eigenvalues are $\pm\sqrt{-1} = \pm i$.
    (c) The eigenvectors are $(1, i)$ and $(1, -i)$.
    (d) The eigenvectors are $(-1, i)$ and $(-1, -i)$.
    (e) None of the above.

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13. Which of the following are projection matrices?
   (a) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
   (b) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
   (c) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
   (d) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
   (e) None of the above.

14. Find the matrix for the reflection of R^2 through the $x = y$ line.
   (a) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
   (b) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
   (c) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
   (d) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
   (e) None of the above.

15. Let A be an $n \times n$ matrix. Which of the following statements are true?
   (a) A symmetric matrix A is positive definite if and only if all the eigenvalues of A are positive.
   (b) A is nonsingular if and only if $\det A=0$.
   (c) A is singular if and only if 0 is an eigenvalue of A.
   (d) $\det A = \lambda_1 + \lambda_2 + \cdots + \lambda_n$, where λ_i's are eigenvalues of A.
   (e) None of the above.

Part II: short answer questions

1. (10 points) Find the limit.
$$\lim_{x\to\infty} \frac{x^{-1/2} - (x+1)^{-1/2}}{x^{-1}}.$$

2. (10 points) Evaluate the integral.
$$\int_0^\infty 2\left(\frac{x}{5}\right)^2 e^{-\left(\frac{x}{5}\right)^2} dx.$$

3. (20 points) Suppose that $u_0 = 1, u_1 = 1$, and $u_n = 2u_{n-1} + 3u_{n-2}$.
   (a) Find eigenvalues of the matrix $A = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}$.
   (b) Find the value of u_{100}.

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