國立臺灣師範大學 109 學年度**碩士班**招生考試試題

科目：**數學基礎**　　　　　　　　　　適用系所：**資訊工程學系**

注意：1.本試題共 3 頁，請依序在答案卷上作答，並標明題號，不必抄題。2.答案必須寫在指定作答區內，否則依規定扣分。

Some related notations:

- We work with column vectors, for example, $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in R^3$.
- The **orthogonal complement** of a nonempty subset S is denoted by $S^{\perp}$.
- The **transpose** of a matrix A is denoted by A^T.
- An $n \times n$ matrix is an **orthogonal matrix** (or **orthogonal**) if its columns form an *orthonormal* basis for R^n.

1. (8 points) Suppose that T: $R^2 \to R^3$ is a linear transformation such that $T\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}$. Determine $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$ for any $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ in R^2.

2. (12 points) For the linear transformation T defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 + x_3 + x_4 \\ 2x_1 - 5x_2 + x_3 + 3x_4 \\ x_1 - 3x_2 + 2x_4 \end{bmatrix}$, (a) find a basis for the range of T; (b) find a basis for the null space of T.

3. (8 points) Given a linear operator T and its characteristic polynomial $f(t)$, determine all the values of the scalar c for which T on R^3 is not diagonalizable, where $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} cx_1 \\ -x_1 - 3x_2 - x_3 \\ -8x_1 + x_2 - 5x_3 \end{bmatrix}$ and $f(t) = -(t-c)(t+4)^2$.

4. (12 points) Let $\boldsymbol{u} = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}$ and W be the solution set of $x_1 - 2x_2 + 3x_3 = 0$.

   (a) Find the orthogonal projection matrix P_W for subspace W.
   (b) Obtain the unique vectors $\boldsymbol{w}$ in W and $\boldsymbol{z}$ in $W^{\perp}$ such that $\boldsymbol{u} = \boldsymbol{w} + \boldsymbol{z}$.

5. (10 points) For the matrix $A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$, find an orthogonal matrix P and a diagonal matrix D such that $P^TAP = D$.

6. (8 points) Express the negation of the proposition "*No student in the class answers this question.*" using quantifiers, and then express the negation in English.

7. (8 points)

(1) Find a recurrence relation for the number of ways to move n bricks if the person moving the bricks can take one brick or two bricks at a time.

(2) What are the initial conditions?

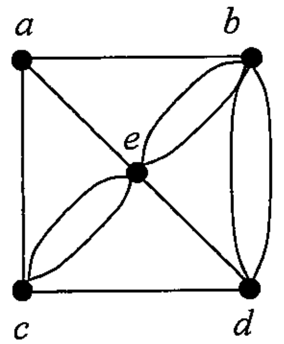
8. (8 points)

(1) Let A = $\{a, c, c, c, e, e, e, e, e\}$, B = $\{e, c, a\}$. Determine whether set A and set B are equal.

(2) Draw the graph of the function $f(x) = \lceil 0.4x \rceil$ from **R** to **R**.

9. (8 points)

(1) Determine whether the following graph has an Euler path and construct such a path if it exists.



(2) Determine whether the following relation R on $\{1, 2, 3, 4\}$ is a partial order.

$R = \{(1, 1), (2, 2), (3, 1), (3, 3), (3, 4), (4, 3), (4, 4)\}$.

10. (8 points) Find an inverse of 55 modulo 102.

11. (10 points) Use strong induction to show that if you can climb on stair or two stairs, and if you can always climb two more stairs once you have climbed a specified number of stairs, then you can climb any number of stairs.