※ 考生請注意：本試題不可使用計算機。　請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

1. (24 pts) **Make** each statement **True** or **False** and **JUSTIFY** each answer. 3 pts for each question. (Correct answer for 2 pts and suitable justification for 1 pts.)

   (a). Suppose that **u**, **v**, and **w** are vectors in $\boldsymbol{R}^n$. If **u** is orthogonal to **v** + **w**, then **u** is orthogonal to **v** and **w**.

   (b). The set of 2×2 matrices that contain exactly two 1's and two 0's is a linearly independent set in $\mathbf{M}_{2\times 2}$.

   (c). If V is a subspace of $\boldsymbol{R}^n$ and W is a subspace of V, then $W^{\perp}$ is a subspace of $V^{\perp}$.

   (d). If **u** is in the row space and the column space of an $n\times n$ matrix **A**, then **u** = **0**.

   (e). If $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ come from different eigenspace of **A**, then it is impossible to express $\mathbf{v}_3$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

   (f). If **A** is diagonalizable and invertible, then $\mathbf{A}^{-1}$ is diagonalizable.

   (g). There is no square matrix **A** such that $\det(\mathbf{A}\mathbf{A}^T) = -1$.

   (h). If $\det(\mathbf{A}) = 0$, then **A** is not expressible as a product of elementary matrices.

2. (21 %) Let **A** be a diagonalizable matrix with characteristic polynomial $p(\lambda) = a_1\lambda^n + a_2\lambda^{n-1} + \cdots + a_{n+1}$.

   (a). If **D** is a diagonal matrix whose diagonal entries are the eigenvalues of **A**, show that $p(\boldsymbol{D}) = a_1\boldsymbol{D}^n + a_2\boldsymbol{D}^{n-1} + \cdots + a_{n+1}\boldsymbol{I} = \mathbf{0}$. (7 pts)

   (b). Show that $p(\mathbf{A}) = 0$. (7 pts)

   (c). Show that if $a_{n+1} \neq 0$, then **A** is invertible and $\mathbf{A}^{-1} = q(\mathbf{A})$ for some polynomial q of degree less than n. (7 pts)

3. (21 %) Let T: $U \to V$ be a linear transformation of vector spaces.

   (a). Prove that if $\ker T = \{\mathbf{0}\}$ then T is one-to-one.　(7 pts)

   (b). Suppose T is one-to-one and $\{\mathbf{u}_1,\ldots,\mathbf{u}_k\}$ is a linearly independent set of vectors in U. Prove that $\{T(\mathbf{u}_1),\ldots,T(\mathbf{u}_k)\}$ is a linearly independent set of vectors in V.　(7 pts)

   (c). Define $U \in \boldsymbol{P}(t)$(a polynomil of degree 2 and its standard form is $\boldsymbol{P}(t) = a_0 + a_1t + a_2t^2$), $\boldsymbol{V} \in \boldsymbol{R}^3$, and $\boldsymbol{T}(U) = \begin{bmatrix} \boldsymbol{P}(-2) \\ \boldsymbol{P}(0) \\ \boldsymbol{P}(2) \end{bmatrix}$. Find U such that the image under T of U is $[11, 1, -1]^T$. (7 pts)

4. (10 %) Find the values of x such that the given matrix is not invertible.

$$A = \begin{bmatrix} 1 & x & x^2 & x^3 \\ a & 1 & x & x^2 \\ b & c & 1 & x \\ p & q & r & 1 \end{bmatrix}$$

5. (24 pts) Let $\boldsymbol{A}$ be the matrix given by $\boldsymbol{A} = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$ with rank $\boldsymbol{A} = 2$.

   a. Compute the $\boldsymbol{A}^{+}$ (the pseudoinverse of $\boldsymbol{A}$, the inverse of reduced singular value decomposition of $\boldsymbol{A}$). (12 pts)

   b. Find a least-squares solution for $\boldsymbol{Ax} = \boldsymbol{b}$, where $\boldsymbol{b} = [1, 0]^T$. (6 pts)

   c. Find the least-squares error for part (c). (6 pts)