※ Please show the detailed calculation process for all questions.

1. (20%) For a random variable x defined on the real number line, denote its probability density function and cumulative distribution function as $f(x)$ and $F_x(\alpha)$, respectively. By definition,

$$\int_{-\infty}^{\infty} f(x)dx = 1,\ F_x(\alpha) = \int_{-\infty}^{\alpha} f(x)dx,$$

and the expectation of any function $g(x)$ can be derived as $E[g(x)] = \int_{-\infty}^{\infty} g(x)f(x)dx$, e.g., for $g(x) = x$, $E[x] = \int_{-\infty}^{\infty} xf(x)dx$.

(a) (10%) Define $H_x(\eta) = \int_{-\infty}^{\eta} F_x(\alpha)d\alpha$. Prove that $H_x(\eta) = E[(\eta - x)^+]$, where $(\eta - x)^+ = \begin{cases} \eta - x & \text{if } \eta - x \geq 0 \\ 0 & \text{if } \eta - x < 0 \end{cases}$. (Hint: you can consider to change the order of integration.)

(b) (10%) Prove that $H_x(\eta) - (\eta - E[x]) = E[(x - \eta)^+]$, where $(x - \eta)^+ = \begin{cases} x - \eta & \text{if } x - \eta \geq 0 \\ 0 & \text{if } x - \eta < 0 \end{cases}$.

2. (30%) The formula for a geometric Asian call option with six parameters (S, X, r, q, σ, T) is as follows.

$$c(S, X, r, q, \sigma, T) = Se^{(a-r)T}N(d_1) - Xe^{-rT}N(d_2),$$

where

$$a = \frac{1}{2}\left(r - q - \frac{\sigma^2}{6}\right),\ d_1 = \frac{\ln(S/X) + \frac{1}{2}\left(r - q + \frac{\sigma^2}{6}\right)T}{\sigma\sqrt{T/3}},\ d_2 = d_1 - \sigma\sqrt{T/3},$$

and $N(\cdot)$ is the cumulative distribution function of the standard normal distribution defined as

$$N(d) = \int_{-\infty}^{d} n(x)dx = \int_{-\infty}^{d} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx,$$

where $n(\cdot)$ is the probability density function of the standard normal distribution.

(a) (10%) Prove that $Se^{(a-r)T}n(d_1) = Xe^{-rT}n(d_2)$.

(b) (6%) Derive and express $\frac{\partial c}{\partial S}$ as the form of $AN(B)$. What are A and B?

(c) (8%) Derive and express $\frac{\partial d_1}{\partial \sigma}$ and $\frac{\partial d_2}{\partial \sigma}$ as $C + Dd_2$ and $E + Fd_2$, respectively. What are C, D, E, and F?

(d) (6%) Derive and express $\frac{\partial c}{\partial \sigma}$ as the form of $Gn(H)$. What are G and H?

3. (10%) Given $x\cos y + y\cos x = 1$, find $\frac{dy}{dx}$.

4. (10%) Find the area of the region bounded by the two curves $y = x^3 - 6x^2 + 8x$ and $y = x^2 - 4x$.

5. (10%) Find $\int \sin 3x \cos 2x\, dx$.

6. (10%) Determine whether the infinite series

$$\sum_{n=1}^{+\infty} \frac{1}{(n^2+2)^{1/3}}$$

is convergent or divergent.

7. (10%) Determine the interval of convergence of the power series $\sum_{n=1}^{+\infty} n(x-2)^n$.

試題隨卷繳回