國立臺灣師範大學 108 學年度碩士班招生考試試題

科目：**數學基礎** 適用系所：**資訊工程學系**

注意：1.本試題共 3 頁，請依序在答案卷上作答，並標明題號，不必抄題。2.答案必須寫在指定作答區內，否則依規定扣分。

Notations (You may skip over this part):

- Let T be a linear operator on R^n and $\mathcal{B} = \{\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_n\}$ be a basis for R^n. The matrix $[[T(\boldsymbol{b}_1)]_{\mathcal{B}} \quad [T(\boldsymbol{b}_2)]_{\mathcal{B}} \quad \cdots \quad [T(\boldsymbol{b}_n)]_{\mathcal{B}}]$ is called the **matrix representation of T with respect to** $\mathcal{B}$, or the $\mathcal{B}$**-matrix of T**, denoted by $[T]_{\mathcal{B}}$.
- An $n \times n$ matrix is an **orthogonal matrix** (or **orthogonal**) if its columns form an *orthonormal* basis for $\mathfrak{R}^n$.
- A linear operator on $\mathfrak{R}^n$ is called an **orthogonal operator** (or **orthogonal**) if its standard matrix is an orthogonal matrix.

1. (6 points) Suppose that the reduced row echelon form R and three columns of A are given by $R = \begin{bmatrix} 1 & 2 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$, $\boldsymbol{a}_1 = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$, $\boldsymbol{a}_3 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$, and $\boldsymbol{a}_4 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$. Determine the matrix A.

2. (12 points) Given a basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$, $T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = -\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = 2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, and $T\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, determine
   (a) $[T]_{\mathcal{B}}$,
   (b) the standard matrix of T, and
   (c) an explicit formula for $T(\boldsymbol{x})$.

3. (10 points) Find the eigenvalues of linear operator T and determine a basis for each eigenspace, where $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 7x_1 - 10x_2 \\ 5x_1 - 8x_2 \\ -x_1 + x_2 + 2x_3 \end{bmatrix}$

4. (12 points) Let $\boldsymbol{u} = \begin{bmatrix} 8 \\ 0 \\ 2 \end{bmatrix}$ and W be the solution set of $\begin{aligned} x_1 + x_2 - x_3 &= 0. \\ x_1 + 2x_2 + 3x_3 &= 0. \end{aligned}$
   (a) Find the orthogonal projection matrix P_W.
   (b) Obtain the unique vectors $\boldsymbol{w}$ in W and $\boldsymbol{z}$ in $W^{\perp}$ such that $\boldsymbol{u} = \boldsymbol{w} + \boldsymbol{z}$.

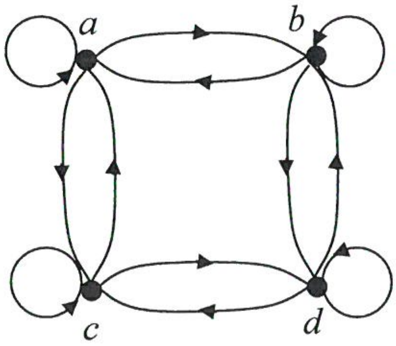
5. (10 points) Find an orthogonal operator T on $\mathfrak{R}^3$ such that $T\left(\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

6. (8 分)

(1) How many partial functions are there from a set with 4 elements to a set with 9 elements?

(2) A group contains n boys and n girls. How many ways are there to arrange these people in a row if the boys and girls alternate?

(3) Determine whether the relation with the directed graph in the flowing figure is an equivalence relation.



(4) Find the in-degree and out-degree of each vertex in the directed graph shown in the above figure.

7. (8 分)

(1) Determine whether the set $S = \{x \mid x \text{ is a bit string not containing the bit } 1\}$ is finite, countably infinite, or uncountable. If the set is countably infinite, exhibit a one-to-one correspondence between the set of positive integers and set S.

(2) Let the value of the floor function at x is denoted by $\lfloor x \rfloor$, and the value of the ceiling function at x is denoted by $\lceil x \rceil$. Draw the graph of the function $f(x) = \lfloor x/2 \rfloor + \lceil x/2 \rceil$.

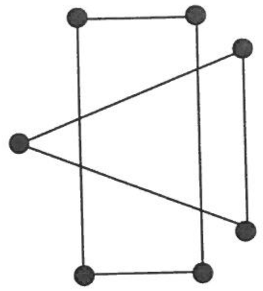
8. (8 分)

(1) The Moon Postal Service (MPS) sells money orders identified by an 11-digit number $x_1 x_2 \ldots x_{11}$. The first ten digits identify the money order; x_{11} is a check digit that satisfies $x_{11} = x_1 + x_2 + \ldots + x_{10}$ **mod** 9. Determine whether the number "88382013456" is a valid MPS money order identification number.

(2) What is the coefficient of x^5y^7 in the expansion of $(2x-3y)^{12}$?

(3) Let $R = \{(2, 2), (3, 2), (4, 3), (5, 4)\}$. Find the powers R^3.

(4) Determine whether the graph below is connected.



9. (6 分) Please show that the following argument is correct.

If today is Monday, I have a test in Digital Logic or Political Science. If my Political Science Professor goes abroad, I will not have a test in Political Science. Today is Monday and my Political Science Professor goes abroad. Therefore I have a test in Digital Logic.

10. (6 分) Find a matrix **A** such that

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ -1 & 1 & 2 \end{bmatrix} \mathbf{A} = \begin{bmatrix} -2 & 5 & 1 \\ 0 & -1 & -1 \\ -1 & 2 & 3 \end{bmatrix}.$$

11. (6 分) For the generating function $(3x-2)^3$, give a closed formula for the sequence it determines.

12. (4 分)

(1) Find 7^{122} **mod** 13.

(2) What is the cardinality of the set $\{x, \{x\}, \{x, \{x\}, \{a\}\}\}$?

13. (4 分) Prove that $1 \cdot 1! + 2 \cdot 2! + \ldots + n \cdot n! = (n+1)! - 1$ whenever n is a positive integer.