國立臺灣師範大學 108 學年度碩士班招生考試試題

科目：**高等微積分**　　適用系所：**數學系**

注意：1.本試題共 2 頁，請依序在答案卷上作答，並標明題號，不必抄題。2.答案必須寫在指定作答區內，否則依規定扣分。

1. (10 分) State and prove the *Bolzano-Weierstrass theorem*.

2. Determine the convergence or divergence for each of the following series:

   (a) (5 分) $\sum_{k=2}^{\infty} \arctan \frac{1}{k}$　　(b) (5 分) $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$

3. Let A be a nonempty subset of $\mathbb{R}^n$. For any point $x \in \mathbb{R}^n$, define the *distance* $\delta_A(x)$ from x to A by

   $$\delta_A(x) := \inf\{\|x-a\| : a \in A\},$$

   where $\|x-a\|$ denotes the Euclidean distance between the two points $x, a \in \mathbb{R}^n$.

   (a) (8 分) Show that δ_A is a continuous function on $\mathbb{R}^n$.

   (b) (7 分) Suppose further that A is a nonempty, closed subset of $\mathbb{R}^n$. Show that $\delta_A(x) = 0$ if and only if $x \in A$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function that satisfies

   $$|f(x) - f(y)| \le |x^2 - y^2| \qquad \text{for all } x, y \in \mathbb{R}.$$

   (a) (5 分) Show that f is a continuous function on $\mathbb{R}$.

   (b) (5 分) Is $f(x)$ differentiable at $x = 0$? If so, find $f'(0)$.

5. (10 分) Let $f : [a,b] \to \mathbb{R}$ be an integrable function. For each $x \in [a,b]$, define

   $$F(x) := \int_x^b f(t)\,dt.$$

   Show that F is uniformly continuous on $[a,b]$.

6. (10 分) Let $I_n = \int_0^{\pi/2} \sin^n x\,dx$ for $n \in \mathbb{N}$. Find the ratio $\frac{I_9}{I_7}$.

7. (10 分) Let f be a bounded, integrable real function on the interval [0,1], and ε be a positive real number. Show that there is an integer $N \in \mathbb{N}$ such that for each integer $n \ge N$, we have

   $$\left| \int_0^1 f(x) x^n dx \right| < \varepsilon.$$

8. Consider the mapping $(u,v,w)=F(x,y,z)$ from $\mathbb{R}^3$ to $\mathbb{R}^3$ defined by

$$\begin{cases} u=x+y+z \\ v=x^2+y^2+z^2 \\ w=x^3+y^3+z^3 \end{cases}.$$

Look at the points $p=(0,1,2)$ and $q=F(0,1,2)=(3,5,9)$ in $\mathbb{R}^3$.

(a) (6 分) Show that there are neighborhoods U of p and V of q such that F has a differentiable inverse mapping G from V to U.

(b) (9 分) Write $(x,y,z)=G(u,v,w)$ from part (a). Compute the partial derivative $\dfrac{\partial x}{\partial u}(q)$.

9. (10 分) Evaluate the double integral $\iint_R \ln(x+y)\mathrm{d}A$, where R is the square region in $\mathbb{R}^2$ whose vertices are the points (1,2), (2,1), (3,2), and (2,3).