國立臺灣師範大學 108 學年度碩士班招生考試試題

科目：**基礎數學**　　　　適用系所：**數學系**

注意：1.本試題共 2 頁，請依序在答案卷上作答，並標明題號，不必抄題。2.答案必須寫在指定作答區內，否則依規定扣分。

Part I : Calculus

一、填充題 (答案本上只寫答案，不需要寫計算過程，請標明題號)

1. (10 分) Find the integrals.

   (a) $\int x \sin(ax)\, dx$, where $a \neq 0$.

   (b) $\int x^2 e^x \, dx$

2. (5 分) Find the volume of the solid generated by revolving the region bounded by the graphs of $y = x^2$ and $y = 4x - x^2$ about the x-axis.

3. (4 分) Find the limits.

   (a) $\lim\limits_{x \to 5^-} \dfrac{\sqrt{25 - x^2}}{x - 5}$　　(b) $\lim\limits_{x \to 0^+} x^{1/x}$

4. (5 分) Find an equation of the tangent line to the graph of $x^2 + xy + y^2 = 4$ at the point $(2, 0)$.

5. (6 分) Find a power series for $\ln(3x^2 + 1)$ centered at $x = 0$ and determine the interval of convergence.

6. (5 分) Find the tangent line to the curve of intersection of the ellipsoid $x^2 + 2y^2 + 2z^2 = 20$ and the paraboloid $x^2 + y^2 + z = 4$ at the point $(0, 1, 3)$.

二、計算題 (請在答案本上寫出計算過程和答案，沒有過程不予計分)

1. (15 分) Evaluate the integral

$$\int_0^2 \int_x^2 x\sqrt{1 + y^3}\, dy\, dx.$$

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Part II : Linear Algebra

1. Let A, B be $n \times n$ matrices over the real numbers.

   (a) (5 points) Show that $\operatorname{rank}(BA) \leq \operatorname{rank}(A)$

   (b) (5 points) Show that $\operatorname{nullity}(A) \leq \operatorname{nullity}(AB)$

2. Let V, W be two subspaces of a vector space U over the real numbers.
   Define $V + W = \{v + w \mid v \in V, w \in W\}$.

   (a) (5 points) Show that $V + W$ is also a subspace of U.

   (b) (10 points) Show that $\dim(V + W) = \dim(V) + \dim(W) - \dim(V \cap W)$.

3. Let $\beta = \{1, 1+x, (1+x)^2, (1+x)^3\}$ be an ordered basis for $P_3(\mathbb{R})$, where $P_3(\mathbb{R})$ is the space of polynomials with real coefficients of degree at most 3. Let T be a linear transformation on $P_3(\mathbb{R})$ whose matrix representation in β is $\begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 2 & -2 & 2 & -2 \\ -2 & 2 & -2 & 2 \end{pmatrix}$.

   (a) (5 points) Prove that two similar matrices have the same eigenvalues.

   (b) (10 points) Find a basis for $P_3(\mathbb{R})$ consisting of eigenvectors of T.

4. (10 points) Find an invertible matrix Q such that $Q^{-1}AQ$ is the Jordan canonical form of
   $A = \begin{pmatrix} 2 & -1 & 0 & 1 \\ 0 & 3 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 3 \end{pmatrix}$.