國立臺灣師範大學 108 學年度碩士班招生考試試題

科目：**數值分析** 適用系所：**數學系**

注意：1.本試題共 1 頁，請依序在答案卷上作答，並標明題號，不必抄題。2.答案必須寫在指定作答區內，否則依規定扣分。

1. Let $g : [a,b] \to [a,b]$ be a differentiable function satisfying $|g'(x)| \le k < 1$ for all $x \in (a,b)$. The Fixed-Point iteration generates a sequence $\{p_n\}_{n=0}^{\infty}$ defined by
$$p_n = g(p_{n-1}), \quad n = 1, 2, \ldots,$$
for any initial point $p_0 \in [a,b]$.
   (a) (5 points) Use the mathematical induction to show that
   $$|p_m - p_{m-1}| \le k^{m-1}|p_1 - p_0|$$
   for all $m \in \mathbb{N}$.
   (b) (10 points) If $p_n \to p$ as $n \to \infty$ and $g(p) = p$ for some $p \in [a,b]$, use the part (a) to deduce that
   $$|p - p_n| \le \frac{k^n}{1-k} \cdot |p_1 - p_0|$$
   for all $n \in \mathbb{N}$.

2. Let $A = \begin{bmatrix} 4 & 3 & 0 \\ 3 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}$.
   (a) (5 points) Is A strictly diagonally dominant? Give your reasons.
   (b) (10 points) Find the Crout factorization of A.
   (c) (5 points) Evaluate the matrix 1-norm for A.
   (d) (5 points) Show that A is symmetric and positive definite.
   (e) (10 points) Find the optimal choice of parameter $\omega > 0$ for the Successive Over-Relaxation (SOR) method performed on A.

3. (5 points) To which zero of $f(x) = (x+2)(x+1)x(x-1)^3(x-2)$ does the Bisection method converge when applied on the interval $[-3, 4]$? Give your reasons.

4. Let $f(x) = x^3 - 6x^2 + 12x - 8$.
   (a) (5 points) Use 3-digit chopping arithmetic to evaluate the approximate vale p_* of $p = f(3.1)$.
   (b) (5 points) How many significant digits does p_* have if $p = 1.3310$?
   (c) (5 points) Describe the Newton's method for solving the root-finding problem $f(x) = 0$.
   (d) (10 points) Use the part (c) to prove that the Newton's method converges linearly to $q = 2$, with asymptotic error constant $2/3$, for any initial guess except q.

5. Let $A \in \mathbb{R}^{n \times n}$ be a nonsingular matrix.
   (a) (10 points) For any vectors $x, y \in \mathbb{R}^n$ with $y^\top A^{-1}x \neq -1$, prove that
   $$(A + xy^\top)^{-1} = A^{-1} - \frac{A^{-1}xy^\top A^{-1}}{1 + y^\top A^{-1}x}.$$
   (b) (10 points) If $\hat{x} \in \mathbb{R}^n$ is a computed solution to the linear system $Ax = b$ with residual $r = b - A\hat{x}$, and $\|\cdot\|$ denotes the matrix (or vector) 2-norm, prove that
   $$\frac{\|x - \hat{x}\|}{\|x\|} \le K_2(A) \cdot \frac{\|r\|}{\|b\|},$$
   where $K_2(A) = \|A\|\|A^{-1}\|$ is the 2-norm condition number of A.