※ 考生請注意：本試題不可使用計算機。　請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

一、True or False $(2\% \times 20 = 40\%)$

(For the following statements, please answer **T** if it is true and **F** otherwise.)

1. Let $\vec{\mathbf{X}}=(X_1,X_2,\ldots\ldots,X_n)'$ be a n-dimensional random vector, n>1, and the components X_i's have common expected value $E(X_i)=\mu$ and variance $Var(X_i)=\sigma^2$. Furthermore, let $T_n=\sum_{i=1}^n X_i$, $\bar{X}_n = T_n/n$, be the summation and mean of X_i's, respectively, and $S_n^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i-\bar{X}_n)^2$ be the sample variance, please answer the following questions:
   i. $E(T_n)=n\mu$ only if X_i's are independent.
   ii. If X_i's are independently distributed as the Poisson distribution, then T_n is a Poisson random variable with $E(T_n)=n\mu$.
   iii. If X_i's are independently distributed as the Exponential Distribution, then $\bar{X}_n$ is an Exponential random variable.
   iv. As in iii, If X_{min} is the minimum of X_i's, the X_{min} is an Exponential random variable.
   v. If $\vec{\mathbf{X}}$ follows a multinomial distribution, then $Cov(X_i,X_j)<0$, $\forall$ i,j.
   vi. If X_i's are normally distributed, then $\vec{\mathbf{X}}$ follows a multivariate normal distribution.
   vii. If X_i's are normally distributed and independent, then $E(\bar{X}_n S_n^2)=\mu\sigma^2$.
   viii. If X_i's are independently and distributed as the standard normal distribution, then $\sum_{i=1}^n X_i^2$ is distributed as a Gamma distribution.
   ix. As in viii, $E(X_i/X_j)=1$.
   x. If $\vec{\mathbf{X}}$ follows a multivariate normal distribution, then the conditional distribution of $X_i|X_j$ is a normal distribution as well.
2. If random variables X and Y have the same moment generating function, then P(X=Y)=1.
3. If A, B are independent events, then A and B^C are independent as well.
4. If two random variables X and Y are independent, then X^2 and Y^2 are also independent.

5. Let $A_1, A_2, \ldots\ldots, A_k$ be a partition of the sample space S and B be an event, then $P(B)=\sum_{i=1}^{k} P(B|A_i)$.

6. Let ν_X and ν_Y be the median of random variable X and Y, then the median of X+Y is $\nu_X + \nu_Y$.

7. Let $K_\Phi(\theta)$ be the power function of a test $\Phi(\vec{X})$, then the type I error rate is the value of $K_\Phi(\theta)$ when $\theta \in N$, where N is the null set.

8. If $E(X^2)=0$, then $E(XY)=0$.

9. It is known that if the MLE $\hat{\theta}$ of the parameter θ is also unbiased, then it is the best unbiased estimator of θ if the complete sufficient statistic exists, therefore the best unbiased estimator of $\tau=\tau(\theta)$ is $\hat{\tau}=\tau(\hat{\theta})$ based in the invariance principle of MLE.

10. $E(T)=\theta$ and Var(T) goes to zero as the sample size increases are required for T to be a consistent estimator of θ.

11. If both of T and R are minimal sufficient for θ, then T and R are invertible functions of each other.

二、Multiple Choice $(6 \times 5\% = 30\%)$

1. Which of the following statement(s) is/are true?
   i. A model with both of the complete sufficient statistic and unbiased estimator must have the UMVUE.
   ii. The multinomial distribution does not have a moment generating function.
   iii. X is a nonnegative random variable, i.e. $X \geqq 0$, then E(X)=0 only if P(X=0)=1.

   (A) i. only (B) ii. only (C) iii. Only (D) i. and ii. (E) i. and iii. (F) ii. and iii. (G) i., ii. , and iii. (H) None

2. Which of the following statement(s) is/are true about a sequence of random vector $\vec{X}_n = (X_{1n}, X_{2n}, \ldots\ldots X_{pn})'$?

   i. If $X_{in} \xrightarrow{d} X_i$, $\forall i=1,\ldots\ldots p$, then $\vec{X}_n \xrightarrow{d} \vec{X}$, $\vec{X} = (X_1, X_2, \ldots\ldots X_p)'$.

   ii. If $X_{in} \xrightarrow{p} a_i$, $\forall i=1,\ldots\ldots p$, then $\vec{X}_n \xrightarrow{p} \vec{a}$, $\vec{a} = (a_1, a_2, \ldots\ldots a_p)'$.

   iii. If $\vec{X}_n \xrightarrow{p} \vec{a}$, then $g(\vec{X}_n) \xrightarrow{p} g(\vec{a})$.

   (A) i. only (B) ii. only (C) iii. Only (D) i. and ii. (E) i. and iii. (F) ii. and iii. (G) i., ii. , and iii. (H) None

3. X is a random variable with distribution function F_X, then

i. F_X may not exists.

ii. F_X is left continuous.

iii. If $a<b$, then $F_X(a)<F_X(b)$.

(A) i. only (B) ii.only (C) iii. Only (D) i. and ii. (E) i. and iii. (F) ii. and iii. (G) i., ii. , and iii. (H) None

4. $\vec{X} = (X_1,X_2,X_3)'$ has moment generating function $M(r,s,t)=(1-r+2s)^{-4}(1-r+4t)^{-3}(1-r)^{-2}$, then

i. X_1 and $(X_2,X_3)'$are independent.

ii. X_2 and X_3 are independent.

iii. If $V=X_1+X_2+X_3$, then the moment generating function of V is $M_V(t) = (1+t)^{-7}(1-t)^{-2}$.

(A) i. only (B) ii.only (C) iii. Only (D) i. and ii. (E) i. and iii. (F) ii. and iii. (G) i., ii. , and iii. (H) None

5. Which of the following statement(s) is(are) correct?

i. The Uniformly Most Powerful test is a Likelihood Ratio Test as well.

ii. The Rao-Blackwell Theorem is used to decrease the bias of an estimator for a more accurate estimation result.

iii. Usually we would use the Type I error rate to evaluate the performance of a test.

(A) i. only (B) ii.only (C) iii. Only (D) i. and ii. (E) i. and iii. (F) ii. and iii. (G) i., ii. , and iii. (H) None

6. Let $X_1,X_2,\ldots,X_n$, $n>1$ be the observations from a normal population with mean θ and variance θ^2, further, let $\bar{X}$ and S^2 be the sample mean and sample variance, respectively, then

i. $(\bar{X}, S^2)$ is the sufficient statistic of θ.

ii. $(\bar{X}, S^2)$ is the minimal sufficient statistic of θ.

iii. $(\bar{X}, S^2)$ is the complete sufficient statistic of θ.

(A) i. only (B) ii.only (C) iii. Only (D) i. and ii. (E) i. and iii. (F) ii. and iii. (G) i., ii. , and iii. (H) None

三、Problems (30%)

1. (10%) Suppose that a rat is in a maze with four possible directions. If it goes in the first direction, it gets out in three minutes. If it chooses the second direction, it returns ti the starting point in five minutes. If it chooses the third direction, with probability 0.2 it returns to the start in four minutes, and with probability 0.8 it returns to the start in seven minutes. If it chooses the fourth direction, it gets out of the maze in six minutes with probability 0. And returns to the maze in eight minutes with probability 0.7. Each time the rat returns it chooses each direction with probability 1/4. What is the expected time until the rat escapes the maze?

2. (10%) Suppose that Xi, i=1,……n, are independently distributed as a common Poisson distribution

with mean θ. Is there an efficient unbiased estimator of θ^2 and why?

3. (10%) Let (X,Y) follow a trinomial distribution with parameter n and (θ_1,θ_2), please find the Likelihood Ratio Test statistic to examine

$$Ha: \theta_1 = \theta_2$$

versus

$$Ho: \theta_1 \neq \theta_2$$