國立臺灣師範大學 107 學年度碩士班招生考試試題

科目：**數學基礎**　　　　　　　　　　適用系所：**資訊工程學系**

注意：1.本試題共 4 頁，請依序在答案卷上作答，並標明題號，不必抄題。2.答案必須寫在指定作答區內，否則依規定扣分。

Notations (You may skip over this part):

- We work with column vectors and denote the set of all column vectors with n component by R^n. For example, $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in R^3$.
- Let A be an $m \times n$ matrix. The function T_A: $R^n \to R^m$ defined by $T_A(\mathbf{x})=A\mathbf{x}$ for all $\mathbf{x}$ in R^n is called the **matrix transformation induced** by A.
- An orthogonal set that is also a basis for a subspace of R^n is called an **orthogonal basis** for the subspace. A basis that is also an orthonormal set is called an **orthonormal basis**.

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1. (8 分) Find the rank and nullity of the matrix $\begin{bmatrix} 1 & 0 & 1 & -1 & 6 \\ 2 & -1 & 5 & -1 & 7 \\ -1 & 1 & -4 & 1 & -3 \\ 0 & 1 & -3 & 1 & 1 \end{bmatrix}$.

2. (10 分) Suppose that T: $R^3 \to R^3$ is a linear transformation such that

$$T\left(\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},\ T\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \text{ and } T\left(\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}.$$ Determine $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right)$ for any $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in R^3.

3. (10 分) Find a 3×3 matrix having eigenvalues 3, 2, 2 with corresponding eigenvectors $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, respectively.

4. (10 分) Let W be the span of $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$, where $\boldsymbol{u_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\boldsymbol{u_2} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, and $\boldsymbol{u_3} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$ are linearly independent vectors in R^4. Apply the Gram-Schmidt process to S to obtain an orthogonal basis S' for W.

5. (12 分) For symmetric matrix $A = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}$,

(a) find an orthonormal basis of eigenvectors and their corresponding eigenvalues.

(b) Use this information to obtain a spectral decomposition of A.

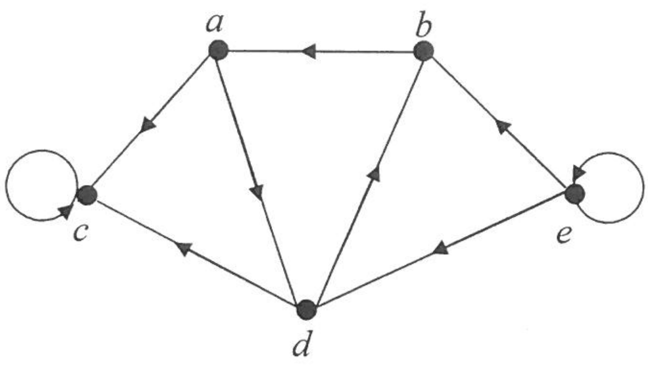
6. (8 分)

(a) Suppose that the universal set is U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}. Express the set {3, 3, 5, 8} with bit strings where the ith bit in the string is 0 if i is in the set and 1 otherwise. Note: the first bit of the string is the leftmost bit.

(b) Let f be a function from the set A to the set B. Let S be a subset of B. We define the inverse image of S to be the subset of A whose elements are precisely all pre-images of all elements of S. We denote the inverse image of S by $f^{-1}(S)$, so $f^{-1}(S)=\{ a \in A \mid f(a) \in S \}$.
If f be a function from R^1 to R^1 defined by $f(x) = x^2$. Find $f^{-1}(\{100\})$.

(c) Determine whether the set C={x| x is the odd negative integers} is finite, countably infinite, or uncountable. If the set is countably infinite, exhibit a one-to-one correspondence between the set of positive integers and set C.

(d) A directed graph is self-converse if it is isomorphic to its converse. Determine whether the following graph is self-converse.



7. (8 分)

(a) Determine whether the following proposition is satisfiable.

$$(p\rightarrow q)\wedge(q\rightarrow\neg p)\wedge(\neg p\rightarrow q)\wedge(q\rightarrow p)$$

(b) Let A(x, y) be the statement "x admires y," where the domain for both x and y consists of all people in the world. Use A(x, y), quantifiers and logical connectives (including negations) to express the following statement:

Nobody admires everybody.

8. (8 分)

(a) Give the function $S(m, n)$ from **N**×**N** to **N** as follows.

$$S(m,n)=\begin{cases}2n & \text{if } m=0\\ 0 & \text{if } m\geq 1 \text{ and } n=0\\ 2 & \text{if } m\geq 1 \text{ and } n=1\\ S(m-1,S(m,n-1)) & \text{if } m\geq 1 \text{ and } n\geq 2\end{cases}.$$

Find the value of $S(2, 2)$.

(b) How many functions are there from the set $\{1, 2, 3, \ldots, n\}$, where n is a positive integer, to the set $\{-1, 0, 1\}$?

9. (8 分)

(a) What is the coefficient of $x^{13}y^{12}$ in the expansion of $(x+y)^{25}$?

(b) Determine whether the relation represented by the zero−one matrix $\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ is a partial order.

(c) Let R be the relation represented by the matrix

$$M_R = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Find the matrix representing $\overline{R}$.

(d) Determine whether the function f from $\{a, b, c, d\}$ to $\{1, 2, 3, 4, 5, 6, 7\}$ with $f(a) = 6$, $f(b) = 1$, $f(c) = 3$, and $f(d) = 4$ is one-to-one.

10. (6 分) Find all solutions to the system of congruences $x \equiv 3$ (**mod** 6), and $x \equiv 4$ (**mod** 11).

11. (6 分) For the generating function $(3x-1)^3$, give a closed formula for the sequence it determines.

12. (6 分) Suppose that

$$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix},$$

where a and b are real numbers. Show that

$$A^n = \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix}$$

for every positive integer n.