國立臺灣師範大學 107 學年度碩士班招生考試試題

科目：高等微積分　　　　　　　　　　　　適用系所：數學系

注意：1.本試題共 1 頁，請依序在答案卷上作答，並標明題號，不必抄題。2.答案必須寫在指定作答區內，否則依規定扣分。

1. (10 points) Does there exist a continuous function $f : [0,1] \to \mathbb{R}$ such that $f([0,1]) = \mathbb{R}$? Justify your answer.

2. (10 points) Let $f : [a,b] \to \mathbb{R}$ be continuous and $c \in [a,b]$. If $f(c) > 0$, show that there exists a closed interval $[a', b'] \subset [a,b]$ on which f is positive.

3. (10 points) Let $f : \mathbb{R} \to \mathbb{R}$ be of C^n. If $f(x_0) = f(x_1) = \cdots = f(x_n)$ where $x_0 < x_1 < \cdots < x_n$, show that there exists $c \in (x_0, x_n)$ such that $f^{(n)}(c) = 0$.

4. (10 points) Find the limit $\displaystyle\lim_{n\to\infty} \int_0^{1/2} \sin(x^n)\,\mathrm{d}x$.

5. (10 points) Indicate why the following calculation is incorrect:
$$\int_{-1}^{2} \frac{1}{x}\,\mathrm{d}x = \ln|x|\Big|_{x=-1}^{x=2} = \ln 2.$$
Correct it.

6. (10 points) Let $E = \left\{ \dfrac{1}{n} \;\middle|\; n \in \mathbb{N} \right\}$ be a subset of $\mathbb{R}$.
   (a) Find a point $p \in \mathbb{R} \setminus E$ such that $E \cup \{p\}$ is compact.
   (b) Show that if a subset C of E is compact, then C must be a finite set (or empty).

7. (10 points) Let the function f be defined on $\mathbb{R}^2 \setminus \{(0,0)\}$ by
$$f(x,y) = \frac{x^2 y}{x^4 + y^2}, \qquad (x,y) \neq (0,0).$$
Does $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ exist? Justify your answer.

8. (10 points) Let $r = r(x,y,z) = \sqrt{x^2+y^2+z^2}$ be the function defined on $\mathbb{R}^3$. Compute $\dfrac{\partial^2 r}{\partial x^2} + \dfrac{\partial^2 r}{\partial y^2} + \dfrac{\partial^2 r}{\partial z^2}$.

9. (10 points) Find all extrema of the function $f(x,y,z) = xy$ subject to the constraints $x^2+y^2+z^2 = 1$ and $2x+y+z = 0$.

10. (10 points) Let E be a nonempty Jordan region in $\mathbb{R}^n$ and $x_0 \in E$. If $f : E \to \mathbb{R}$ is integrable and continuous at x_0, prove that
$$\lim_{r\to 0+} \frac{1}{\mathrm{Vol}(B_r(x_0))} \int_{B_r(x_0)} f\,\mathrm{d}V = f(x_0).$$
(Remark. $B_r(x_0)$ is the open ball in $\mathbb{R}^n$ centered at x_0 with radius r, and $\mathrm{Vol}(B_r(x_0))$ is its n-dimensional volume.)