國立臺灣師範大學 107 學年度碩士班招生考試試題

科目：**基礎數學**　　　　　　　　　　適用系所：**數學系**

注意：1.本試題共 2 頁，請依序在答案卷上作答，並標明題號，不必抄題。2.答案必須寫在指定作答區內，否則依規定扣分。

Part I : Calculus

1. (12 points) Evaluate the integrals:
   (a) $\int_0^2 x|x-1|dx$; (b) $\int \frac{12x^3-24}{\sqrt{x^4-8x+5}}dx$;
   (c) $\int \sqrt[4]{\tan x}\sec^2 x dx$; (d) $\int \tan x dx$.

2. (3 points) Find a power series for $\frac{1}{1+x^3}$, centered at 0.

3. (3 points) Let $\frac{dy}{dx} = xe^{9x^2}$. Find y.

4. (4 points) Describe the definition of the series. Prove that the series
$$\sum_{n=1}^{\infty} \frac{1}{1+2+3+\cdots+n}$$
converges.

5. (4 points) Describe the definition of an improper integral. Assume that f' is continuous on $[0,\infty)$ and $\lim_{x\to\infty} f(x) = 0$. Evaluate
$$\int_0^{\infty} f'(x)dx.$$

6. (9 points) Evaluate the derivatives: (a) $\frac{d}{dx}\left[\int_0^x \sin(t^2)dt\right]$, (b) $\frac{d}{dx}\left[\int_{x^2}^{\sin x} \sin(t^2)dt\right]$, and (c) $\frac{d}{dx}\left[\int_{x^2}^{\sin x} x\sin(t^2)dt\right]$.

7. (3 points) The function $g(x) = 0$ if x is rational and $g(x) = x$ if x is irrational. Prove that
$$\lim_{x\to 0} g(x) \text{ exists.}$$

8. (3 points) Find the volume of the solid of revolution formed by revolving the region bounded by $y = \frac{1}{\sqrt{1+x^2}}$, $y = 0$, $x = -1$ and $x = 1$ about the x-axis.

9. (4 points) Find the volume of the solid region bounded above by the hemisphere
$$z = \sqrt{9-x^2-y^2}$$
and below by the circular region R given by $x^2+y^2 \le 4$.

10. (5 points) Let R be the region bounded by the lines $x-2y=0$, $x-2y=-6$, $x+y=5$ and $x+y=1$. Evaluate the double integral
$$\iint_R 3xy dA.$$

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Part II : Linear Algebra

11. Let V be the vector space of all polynomials with real coefficients and of degree at most 3, together with the zero polynomial. Consider the linear transformation T from V to itself defined by

$$T(p(x)) = (x-1)p'(x) + 2p''(x), \text{ for all } p(x) \in V.$$

(a) (8 points) Let $\beta = \{1, x+1, (x+1)^2, (x+1)^3\}$ be an ordered basis for V. Find the matrix representation of T relative to β.

(b) (7 points) Find the range and the kernel of T.

12. (10 points) Find an orthonormal basis consisting of eigenvectors of the matrix

$$A = \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}.$$

13. (12 points) Prove that

(a) Eigenvectors of a real matrix that correspond to different eigenvalues are linearly independent.

(b) Eigenvectors of a real symmetric matrix that correspond to different eigenvalues are orthogonal.

14. Let A be a real matrix of size $m \times n$. Let $\mathrm{N}(A)$ and $\mathrm{R}(A)$ denote the null space and the row space of A, respectively.

(a) (8 points) Prove that $\mathbb{R}^n$ can be written as a direct sum of $\mathrm{N}(A)$ and $\mathrm{R}(A)$.

(b) (5 points) Let f be a function on $\mathrm{R}(A)$ given by $f(x) = Ax$. Prove that f is a one-to-one function onto the column space of A.