國立臺灣師範大學 107 學年度碩士班招生考試試題

科目：線性代數與代數　　　　適用系所：數學系

注意：1.本試題共 2 頁，請依序在答案卷上作答，並標明題號，不必抄題。2.答案必須寫在指定作答區內，否則依規定扣分。

Part I : Linear Algebra

1. (10 points)

   (a) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & t \\ 1 & 4 & t^2 \end{bmatrix}$, t is a real number. Find the possible rank of A.

   (b) Let A be a 7×3 matrix. If the null space of A is spanned by $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$. Find rank$(A)$.

2. (10 points) Let A and C be matrices over real numbers such that the product AC is defined. Prove that $\text{rank}(AC) \leq \min\{\text{rank}(A), \text{rank}(C)\}$.

3. (15 points) Let U, V be finite dimensional subspace of a vector space W over the real number.

   (a) Let $U + V = \{u + v \mid u \in U \text{ and } v \in V\}$. Show that $U + V$ is a subspace of W.

   (b) Show that $\dim(U + V) = \dim(U) + \dim(V) - \dim(U \cap V)$.

4. (15 points) Let V be the vector space of all $n \times n$ matrices over real numbers. For a matrix A in V, let $T_A : V \to V$ be the linear transformation defined by $T_A(B) = AB,\ B \in V$.

   (a) Show that T_A is invertible if and only if A is invertible.

   (b) Show that T_A and A have the same eigenvalues.

Part II : Algebra

In the following, $\mathbb{N}$ is the set of all positive integers and $\mathbb{Z}$ is the set of all integers.

1. (7 points) Let G be a cyclic group of order 2018 generated by element $a \in G$. Let
$$\varphi : G \to G \text{ be given by } \varphi(g) = g^5 \text{ for } g \in G.$$
It is a fact (and easy to show) that φ is an automorphism of G. Find a positive integer k such that $a^k = \varphi^{-1}(a)$ where φ^{-1} is the inverse of φ.

2. (8 points) Let H and K be subgroups of the group G such that $hk = kh$ for $h \in H$ and $k \in K$. Let $N = H \cap K$ and let
$$\overline{N} = \{(n, n^{-1}) \mid n \in N\} \subset H \times K.$$
Show that $\overline{N}$ is a normal subgroup of $H \times K$ and $HK \simeq (H \times K)/\overline{N}$.

3. (15 points) Recall that A_n $(n \geq 2)$ is the alternating group on n letters.
   (a) Does A_5 has a subgroup of order 30? Explain your answer.
   (b) Determine the number of Sylow 3-subgroups and Sylow 5-subgroups of A_5. The same question for S_5 (the symmetric group on 5 letters).

4. (10 points) Let $m > 1$ be a positive integer and let $k = \mathbb{Z}/\langle m\rangle$ be the factor ring of $\mathbb{Z}$ by the ideal $\langle m\rangle = \{km \mid k \in \mathbb{Z}\}$ generated by m. Let $f(x) \in k[x]$ be a polynomial of degree $d \geq 1$ over k and let $R = k[x]/\langle f(x)\rangle$ where $\langle f(x)\rangle = \{g(x)f(x) \mid g(x) \in k[x]\}$ is the ideal generated by $f(x)$.
   (a) How many elements does R have? You need to explain your answer.
   (b) Prove that R is an integral domain if and only if m is a prime number and $f(x)$ is irreducible in $k[X]$.

5. (10 points) Recall that the characteristic of a ring R is the positive integer
$$\mathrm{char}(R) = \min\{n \in \mathbb{N} \mid n \cdot r = 0_R \ \forall\, r \in R\} \quad (0_R \text{ is the zero element of } R)$$
provided that the minimum exists; otherwise we set $\mathrm{char}(R) = 0$.
   (a) Show that if R is a finite ring of order n then $\mathrm{char}(R) > 0$ and $\mathrm{char}(R) \mid n$.
   (b) Compute the characteristic of the ring $\mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_k}$ where $k \geq 1$ and $m_1, \ldots, m_k \in \mathbb{N}$. You need to explain your answer.