INSTRUCTIONS: This paper consists of three sections. Section A consists of 20 multiple choice (MC) questions. Section B consists of 10 MC questions asking about the logical implications of two statements. Each MC question carries 2 marks. Section C consists of 3 questions. Each carries 20 marks. You only have to answer two of them.

SECTION A: Multiple Choice (General)

Question 1

Let $\mathcal{D} = \{1, 2, 4, 6, 7\}$ and $\mathcal{R} = \{3, 4, 5\}$ be the domain and range of a mapping $\mathcal{M}$ defined as follows :

$$\mathcal{M} = \{(1,4), (2,3), (4,4), (6,3), (7,3)\},$$

i.e. $\mathcal{M}(1) = 4$, $\mathcal{M}(2) = 3$, $\mathcal{M}(4) = 4$, $\mathcal{M}(6) = 3$, $\mathcal{M}(7) = 3$. What is the property of this mapping?

Answer:

(a) $\mathcal{M}$ is an injective mapping.

(b) $\mathcal{M}$ is a surjective mapping.

(c) $\mathcal{M}$ is a bijective mapping.

(d) None of the above.

Question 2

Which of the following statements about e is(are) true ?

(i) $e = \lim_{n\to\infty} (1 + (1/n))^n$;

(ii) $e = \lim_{h\to 0} (1 + h)^{1/h}$;

(iii) $\ln(e^2) = 2$.

Answers :

(a) (i) and (ii) only

(b) (ii) and (iii) only;

(c) (i) and (iii) only;

(d) (i), (ii) and (iii).

Question 3

The motion of a particle whose position $P(x, y)$ at time t is given by

$$x = a\cos(t), \quad y = b\sin(t),$$

for $0 \le t \le 2\pi$. Find the line tangent to the curve at the point $(a/\sqrt{2}, b/\sqrt{2})$, where $t = \pi/4$. (The constants a and b are both positive.)

(a) $y = -\frac{b}{a}x + \sqrt{2}b$.

(b) $y = \frac{b}{a}x - \sqrt{2}b$.

(c) $y = -\frac{a}{b}x + \sqrt{2}a$.

(d) $y = \frac{a}{b}x - \sqrt{2}a$.

Question 4

Which of the following statements are true?

(i) If $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ exist, $\int_a^b f(x) + g(x)dx$ exists.

(ii) If $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ exist, $\int_a^b f(x) - g(x)dx$ exists.

(iii) If $\int_a^b f(x) + g(x)dx$ exists, both $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ exist.

Answer:

(a) (i) and (ii) only.

(b) (ii) and (iii) only.

(c) (i) and (iii) only.

(d) (i), (ii) and (iii).

Question 5

Which of the following statements are true?

(i) If $f'(a)$ and $g'(a)$ exist, $f'(a) + g'(a)$ exists.

(ii) If $f'(a)$ and $g'(a)$ exist, $f'(a) - g'(a)$ exists.

(iii) If $\frac{d}{dx}(f(x) + g(x))\big|_{x=a}$ exists, both $f'(a)$ and $g'(a)$ exist.

Note that $f'(a) = \frac{d}{dx}f(x)\big|_{x=a}$.

Answer:

(a) (i) and (ii) only.

(b) (ii) and (iii) only.

(c) (i) and (iii) only.

(d) (i), (ii) and (iii).

Question 6

Which of the following statements are true?

(i) If $f(x)$ is injective, $f^{-1}(x)$ is injective.

(ii) If $f(x)$ is surjective, $f^{-1}(x)$ is surjective.

(iii) If $f(x)$ is bijective, $f^{-1}(x)$ is bijective.

Here $f^{-1}(x)$ is the inverse mapping of $f(x)$.

Answer:

(a) (i) only.

(b) (ii) only.

(c) (iii) only.

(d) (i) and (ii) only.

(e) (ii) and (iii) only.

(f) (i) and (iii) only.

Question 7

Let $f(x)$ is a differentiable function and $|f(x) - f(y)| \leq K|x - y|$ for all $x, y \in R$ and K is a positive constant. Which of the following statements abut $f(x)$ are true?

(i) $df/dx \leq K$ for all $x \in R$.

(ii) $|df/dx| \leq K$ for all $x \in R$.

(iii) $\int_0^a df(x) \leq Ka$.

Answer:

(a) (i) and (ii) only.

(b) (ii) and (iii) only.

(c) (i) and (iii) only.

(d) (i), (ii) and (iii).

Question 8

Suppose

$$I(\alpha) = \int_{a(\alpha)}^{b(\alpha)} f(x, \alpha) dx,$$

where $f(x, \alpha)$ is integrable function of x it the range $a \leq x \leq b$, a and b being continuous and at least once differentiable functions of α. Which of the following statements are true?

(i)

$$\frac{dI(\alpha)}{d\alpha} = f(b, \alpha)\frac{db}{d\alpha} - f(a, \alpha)\frac{da}{d\alpha} + \int_a^b \frac{\partial f(x, \alpha)}{\partial \alpha} dx.$$

(ii)

$$\frac{dI(\alpha)}{d\alpha} = f(b, \alpha)\frac{db}{d\alpha} + f(a, \alpha)\frac{da}{d\alpha} + \int_a^b \frac{\partial f(x, \alpha)}{\partial \alpha} dx.$$

(iii) If a and b do not depend on α,

$$\frac{dI(\alpha)}{d\alpha} = \int_a^b \frac{\partial f(x, \alpha)}{\partial \alpha} dx.$$

Answer:

(a) (iii) only.

(b) (i) and (ii) only.

(c) (ii) and (iii) only.

(d) (i) and (iii) only.

Question 9

Let $F(x,y,z)$ and $G(x,y,z)$ are differentiable functions of x, y and z. Moreover, $y(x)$ and $z(x)$ are functions of x. Besides,

$$F(x,y,z)=0,\ \ G(x,y,z)=0.$$

Which of the following statements are true?

(i)

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\frac{dy}{dx}+\frac{\partial F}{\partial z}\frac{dz}{dx}=0.$$

(ii)

$$\frac{dy}{dx}=-\left(\frac{\partial F}{\partial x}\frac{\partial G}{\partial z}-\frac{\partial F}{\partial z}\frac{\partial G}{\partial x}\right)\left(\frac{\partial F}{\partial y}\frac{\partial G}{\partial z}-\frac{\partial F}{\partial z}\frac{\partial G}{\partial y}\right)^{-1}.$$

(iii)

$$\frac{dy}{dx}=\left(\frac{\partial F}{\partial x}\frac{\partial G}{\partial y}-\frac{\partial F}{\partial y}\frac{\partial G}{\partial x}\right)\left(\frac{\partial F}{\partial y}\frac{\partial G}{\partial z}-\frac{\partial F}{\partial z}\frac{\partial G}{\partial y}\right)^{-1}.$$

Answer:

(a) (i) and (ii) only.

(b) (ii) and (iii) only.

(c) (i) and (iii) only.

(d) (i), (ii) and (iii).

Question 10

Given that $p(x,y)$ is an unknown joint probability density function defined on $(x,y)\in[a,b]^2$. The marginal probability density functions $p(x)$ and $p(y)$ are unknown. The only information is that the conditional probabilities $p(x|y)$ and $p(y|x)$ are known and they are larger than zero. Is it possible to find the joint probability density function $p(x,y)$?

Answer:

(a) No, it is not possible.

(b) Yes, it is possible. The answer is that

$$p(x,y)=p(y|x)\left(\int_a^b \frac{p(y|x)}{p(x|y)}dy\right)^{-1}.$$

(c) Yes, it is possible. The answer is that

$$p(x,y)=p(y|x)\left(\int_a^b \frac{p(x|y)}{p(y|x)}dy\right)^{-1}.$$

(d) Yes, it is possible. The answer is that

$$p(x,y)=p(x|y)p(y).$$

Question 11

It can readily be shown that, for $\alpha>0$,

$$\int_0^\infty \exp(-\alpha x)dx = \frac{1}{\alpha},$$
$$\int_0^\infty \exp(-x^2)dx = \frac{\sqrt{\pi}}{2}.$$

Which of the following statements are true?

(i) $$\int_0^\infty x^n\exp(-\alpha x)dx=\frac{n!}{\alpha^{n+1}}.$$

(ii) $$\int_0^\infty \exp(-\alpha x^2)dx=\frac{1}{2}\left(\frac{\pi}{\alpha}\right)^{1/2}.$$

(iii) $$\int_0^\infty x^{2n}\exp(-\alpha x^2)dx =\frac{1\times3\times5\times\cdots\times(2n-1)\sqrt{\pi}}{2^{n+1}\alpha^{n+1/2}},$$

where $n\geq 1$.

Answer:

(a) (i) and (ii) only.

(b) (ii) and (iii) only.

(c) (i) and (iii) only.

(d) (i), (ii) and (iii).

Question 12

Suppose $f(x)$ is an odd function and $g(x)$ is an even function. Which is the following statements is true?

(i) $\int_{-a}^{a} f(-x)g(-x)dx=0.$

(ii) $\int_{-a}^{a} f(-x)g(x)dx=0.$

(iii) $\int_{-a}^{a} f(x)g(-x)dx = 0$.

Answer:

(a) (i) and (ii) only.

(b) (ii) and (iii) only.

(c) (i) and (iii) only.

(d) (i), (ii) and (iii).

Question 13

What is the value of the following definite integral?

$$F(a,b) = \int_0^1 x^a(1-x)^b dx,$$

where a and b are positive integers.

(a) $F(a,b) = \frac{a!b!}{(a+b)!}$.

(b) $F(a,b) = \frac{a!b!}{(a+b+1)!}$.

(c) $F(a,b) = \frac{(a+b)!}{a!b!}$.

(d) $F(a,b) = \frac{(a+b+1)!}{a!b!}$.

Question 14

A point (x,y) is moving on a plane in accordance with the following continuous functions $x(t)$ and $y(t)$,

$$x(t) = \sin(t), \quad y(t) = \cos(t).$$

$(x(t), y(t))$ is the location of a point on a plane at time t. What is the total length the point was moving from $t = \pi/4$ to $t = \pi$?

Answer:

(a) No answer.

(b) $\pi/2$.

(c) $3\pi/4$.

(d) 2π.

Question 15

Suppose $f(x)$ is a continuous function defined on $[-\pi,\pi]$. Which of the following statements are true?

Answer:

(a) $\inf_{-\pi}^{\pi} f(x)dx = \inf_{-\pi}^{\pi} f(\pi - x)dx$ only if $f(x)$ is an odd function.

(b) $\inf_{-\pi}^{\pi} f(x)dx = \inf_{-\pi}^{\pi} f(\pi - x)dx$ only if $f(x)$ is an even function.

(c) $\inf_{-\pi}^{\pi} f(x)dx = \inf_{-\pi}^{\pi} f(\pi - x)dx$ only if $f(x)$ is differentiable function.

(d) $\inf_{-\pi}^{\pi} f(x)dx = \inf_{-\pi}^{\pi} f(\pi - x)dx$ for all continuous function $f(x)$.

Question 16

Which of the following statements are correct if $f(x)$ is an even function?

(i) $\int_0^b f(x-a)dx = \int_0^b f(a-x)dx$, where $b > 0$ and $a \in [0,b]$.

(ii) $f(ax) = af(-x)$ for all $a \geq 0$.

(iii) $f(x-a) - f(a-x) = 0$ for all $a \geq 0$.

Answer:

(a) (i) and (ii) only.

(b) (ii) and (iii) only.

(c) (i) and (iii) only.

(d) (i), (ii) and (iii).

Question 17

Suppose $f(x)$ is an odd function and $F(x)$ is an integral defined upon $f(x)$.

$$F(x) = \int_{-x}^{x} f(u)du.$$

Which of the following statements are true?

(a) $dF(x)/dx = f(x)$.

(b) $dF(x)/dx = 2f(x)$.

(c) $dF(x)/dx = -f(x)$.

(d) $dF(x)/dx = 0$.

Question 18

Which of the following conditions is a necessary and sufficient condition for applying mean value theorem?

(a) $f(x)$ must be a continuous function.

(b) $f(x)$ must be a differentiable function.

(c) $f(x)$ must be an integrable function.

(d) None of the above.

Question 19

Suppose $f(x) = \exp(a + b\ln(1+x))$, where a and b are positive integers larger than one. What is the value of the following definite integral?

$$\int_{-1}^{b} f(x)dx.$$

(a) $(1+a)^{1+a}\exp(b)$.

(b) $(1+b)^{1+b}\exp(a)$.

(c) $(1+a)^{a}\exp(b)$.

(d) $(1+b)^{b}\exp(a)$.

Here C_0 is a constant.

Question 20

Given a scalar function $V(\mathrm{x})$ of $\mathrm{x} \in R^n$,

$$\mathrm{x} = (x_1, x_2, \cdots, x_n)^T,$$

and

$$V(\mathrm{x}) = \mathrm{x}^T \mathbf{W} \mathrm{x},$$

where $\mathbf{W}$ = is a $n \times n$ symmetric matrix, i.e. $\mathbf{W} = \mathbf{W}^T$, and its elements are denoted as w_{ij} for all $i, j = 1, \cdots, n$. Let x_{-i} be the vector in which $x_i = 0$, i.e.

$$\mathrm{x}_{-i} = (x_1, \cdots, x_{i-1}, 0, x_{i+1}, \cdots, x_n)^T.$$

Here, the superscript T refers to transpose. Which of the following statement is true ?

(i) $V(\mathrm{x}) - V(\mathrm{x}_{-i}) = 2\sum_{j=1}^{n} w_{ij}x_i x_j$.

(ii) $V(\mathrm{x}) - V(\mathrm{x}_{-i}) = 2\sum_{j=1, j\neq i}^{n} w_{ij}x_i x_j + w_{ii}x_i^2$.

Answer :

(a) None of them.

(b) (i) only

(c) (ii) only

(d) All of them.

SECTION B: Multiple Choice (Logical)

Instructions for Question 21 - Question 30 : *In each question, two statements X and Y are given. You have to identify the logical implication between X and Y. You have to give answer either one of the following options.*

(a) Both statements have no logical implication or at least one of the statements is false.

(b) X implies Y, i.e. $X \Rightarrow Y$, only.

(c) Y implies X, i.e. $X \Leftarrow Y$, only.

(d) Both X implies Y and Y implies X, i.e. $X \Leftrightarrow Y$.

For example "$X \Rightarrow Y$", it means that the statement Y is true if statement X is true. In other words, the proof of Y can be accomplished by using the statement X.

Question 21

X: V is a vector space.

Y: V is a group.

Question 22

In the following statements, $|x|$ refers to the absolute value.

X: For all $x_1, x_2 \in R$, $|x_1 + x_2| \leq |x_1| + |x_2|$.

Y: For all $x_1, x_2 \in R$, $|x_1 - x_2| \geq |x_1| - |x_2|$.

Question 23

X: For all $x_1, x_2 \in R$, $(x_1 - x_2)^2 \geq 0$.

Y: For all $x_1, x_2 \in R$, $x_1 x_2 \leq (x_1^2 + x_2^2)/2$.

Question 24

Let $\{\mathbf{a}_n\}_{n=1}^{\infty}$ be a sequence of real vectors, $\mathbf{a}_n = (x_n, y_n)^T$ for $n \geq 1$. The magnitude of $\mathbf{a}_n$ is denoted as $\|\mathbf{a}_n\|$ and it is defined as follows :

$$\|\mathbf{a}_n\| = \sqrt{x_n^2 + y_n^2}.$$

X: $\lim_{n\to\infty} \mathbf{a}_n$ exists.

Y: $\lim_{n\to\infty} \|\mathbf{a}_n\|$ exists.

Question 25

For a sequence of real numbers $\{a_n\}_{n=1}^{\infty}$, in which $a_{n+1} = 0.8a_n + 1$ and $a_1 = 1$.

X: $\lim_{n\to\infty} a_n = 5$.

Y: $a_n \leq 5$ and $a_{n+1} \geq a_n$ for all n.

Question 26

X: $f(x)$ is differentiable in $[a, b]$.

Y: $f(x)$ is integrable in $[a, b]$.

Question 27

X: Both $\lim_{n\to\infty}(a_n + b_n)$ and $\lim_{n\to\infty}(a_n - b_n)$ exist.

Y: Both $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist.

Question 28

X: Both $\lim_{n\to\infty} a_n b_n$ and $\lim_{n\to\infty} a_n/b_n$ exist.

Y: Both $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist.

Question 29

X: $f(x)$ is differentiable in $[a, b]$.

Y: $f(x)$ is a continuous function in $[a, b]$ and there exists $\xi \in [a, b]$ such that

$$f'(\xi) = \frac{f(a) - f(b)}{a - b}.$$

Question 30

X: $\int_0^\infty \exp(-\alpha x)dx = 1/\alpha$.

Y: $\int_0^\infty \exp(-\alpha x^2)dx = \sqrt{\pi/\alpha}$.

SECTION C: Short Questions

Remind that you only need to answer two questions in this section.

Question 31

Given a function $f(x)$ defined as follows :

$$f_T(x) = \frac{1}{1 + \exp(-x/T)}, \quad (1)$$

where T is a positive constant. It has been shown by David C. Haley in 1952 that a cumulative normal distribution could be approximated by the above function, i.e.

$$\int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz \approx \frac{1}{1 + \exp(-\gamma x)}, \quad (2)$$

where $\gamma = 1.702$.

(a) Show that

$$f_T(x + \delta) \approx \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi S_T}} \exp\left(-\frac{(z+\delta)^2}{2S_T}\right) dz,$$

where $S_T = \gamma^2 T^2$. [4 marks]

(b) Show that

$$\begin{aligned} &\frac{\delta^2}{S_N} + \frac{(\delta + z)^2}{S_T} \\ = &\left(\frac{S_N + S_T}{S_N S_T}\right)\left(\delta + \frac{S_N}{S_N + S_T} z\right) \\ &+ \frac{z^2}{S_N + S_T}. \end{aligned}$$

[4 marks]

(c) Given that δ is a normal distributed random variable, i.e.

$$p(\delta) = \frac{1}{\sqrt{2\pi S_N}} \exp\left(-\frac{\delta^2}{2S_N}\right),$$

where S_N is a positive constant. By using the results in (a) and (b) together with the approximation (2), show that

$$\int_{-\infty}^{\infty} f_T(x+\delta)p(\delta)d\delta \approx f_T\left(\frac{x}{\alpha}\right),$$

where

$$\alpha = \sqrt{1+\frac{S_N}{S_T}}.$$

That is to say,

$$\int_{-\infty}^{\infty} f_T(x+\delta)p(\delta)d\delta \approx \frac{1}{1+\exp(-x/(\alpha T))}$$

[10 marks]

(d) From (c), find the limit

$$\lim_{T\to 0}\int_{-\infty}^{\infty} f_T(x+\delta)p(\delta)d\delta$$

at $x=0$. [2 marks]

Question 32

(a) Given a sequence of positive numbers, $\{a_1,\cdots,a_n,\cdots\}$, and $a_{n+1} < a_n$ for all n. Show that

$$\lim_{n\to\infty} a_n = 0$$

if $\lim_{n\to\infty}\sum_{k=1}^{n} a_k = K$, where K is a finite number. [8 marks]

(b) Given the following algorithm in which x_k is a random variable sampled from a population in which the population mean is zero and the range of the population is finite, i.e. $-L < x_k < L$. Moreover, $z_0 = 0$.

$$z_{n+1} = z_n - \frac{1}{n+1}(z_n - x_{n+1}).$$

Show that z_n is bounded for all $n \geq 0$. [4 marks]

(c) Let $e_{n+1} = |z_{n+1} - z_n|$. Show that e_n is bounded for all $n \geq 1$. [6 marks]

(d) What is the value of z_n? [2 marks]

Question 33

(a) Given two sets of real numbers $\{a_1,a_2,\cdots,a_n\}$ and $\{b_1,b_2,\cdots,b_n\}$. Show that

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{j=1}^{n} b_j^2\right).$$

[10 marks]

(b) Given two differentiable functions $f(x)$ and $g(x)$ Show that

$$\left(\int_a^b f(x)g(x)dx\right)^2 \leq \left(\int_a^b f(x)^2dx\right)\left(\int_a^b g(x)^2dx\right).$$

[10 marks]