國立臺灣大學 107 學年度碩士班招生考試試題

※ 注意：全部題目均請作答於試卷內之「非選擇題作答區」，請標明題號依序作答。

- Unless otherwise specified, everything is over $\mathbb{R}$.
- The ordinary inner product of $\mathbb{R}^n$ is denoted by $\vec{u} \cdot \vec{v}$.
- $\mathcal{M}_{m\times n}$ is the space of $m \times n$ matrices; $f_M(t) = \det(tI_n - M)$ is the characteristic polynomial of M; $\operatorname{im} A$ is the image of A; $\ker A$ is the kernel of A; $V^{\perp}$ is the normal space of V. Parallelepiped = 平行六面體.
- Duel space V^* of real vector space V is $\{\alpha \mid \alpha : V \to \mathbb{R}, \alpha \text{ is linear}\}$.

A. [15%] 是非題. 若錯誤, 需說明原因或給出反例. <u>本題答案須寫在答案簿最前面</u>.

1. There is a linear transformation $A : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\operatorname{im} A = \ker A$.
2. $A \in \mathcal{M}_{n\times n}$. Suppose $A^2 = A$ then $\ker A = (\operatorname{im} A)^{\perp}$.
3. For any A, B, $C \in \mathcal{M}_{n\times n}$, $\operatorname{tr}(ABC) = \operatorname{tr}(CBA)$.
4. The matrix representation A of an adjoint transformation satisfies $A^{t} = A$.
5. Symmetric matrix A is positive definite if and only if all its diagonal elements are positive.

B. [85%] 計算/證明題。(6A) 和 (6B) 只選擇一題作答，兩題皆答，以先寫者計算。

(1) [15%] Find all Jordan canonical forms for square matrices in $\mathcal{M}_{n\times n}$, $n \le 6$, with minimal polynomial $(t-1)^2(t+1)^2$.

(2) [15%] For A, $B \in \mathcal{M}_{m\times n}$, show that $f_{BA^t}(t) = f_{B^tA}(t) \cdot t^{m-n}$.

(3) [15%] Consider $V = \{A \mid AX = XA, \text{for any } X \in \mathcal{M}_{n\times n}\} \subset \mathcal{M}_{n\times n}$. Show that V is an one dimensional subspace of $\mathcal{M}_{n\times n}$.

(4) [15%] $A \in \mathcal{M}_{n\times n}$. Suppose $(t^2+1) | f_A(t)$, are there $\vec{u}$, $\vec{v} \in \mathbb{R}^n$ such that $A\vec{u} = \vec{v}$ and $A\vec{v} = -\vec{u}$? Prove or disprove it.

(5) [15%] U is a subspace of a finite dimensional vector space V. Consider $D_U \subset V^*$ defined by $\{\alpha \in V^* \mid U \text{ is a subspace of } \ker \alpha\}$. Show that D_U is a subspace of dimension $\dim V - \dim U$.

(6A) [10%] Show the volume V of the parallelepiped span by $\vec{u}$, $\vec{v}$, $\vec{w} \in \mathbb{R}^n$ satisfies

$$\begin{aligned} V^2 =& \|\vec{u}\|^2\,\|\vec{v}\|^2\,\|\vec{w}\|^2 + 2\,(\vec{u}\cdot\vec{v})\,(\vec{v}\cdot\vec{w})\,(\vec{w}\cdot\vec{u}) \\ & - \|\vec{u}\|^2\,(\vec{v}\cdot\vec{w})^2 - \|\vec{v}\|^2\,(\vec{w}\cdot\vec{u})^2 - \|\vec{w}\|^2\,(\vec{u}\cdot\vec{v})^2 \end{aligned}$$

(6B) [10%] Following diagram of vector spaces and linear transformations satisfies

(a) $\ker f_{i+1} = \operatorname{im} f_i$, $\ker g_{i+1} = \operatorname{im} g_i$, $i = 0, 1, 2, 3, 4$.

(b) $\alpha_{i+1} \circ f_i = g_i \circ \alpha_i$, $i = 1, 2, 3, 4$.

$$\begin{array}{ccccccccccccc}
\{0_{A_1}\} & \xrightarrow{f_0} & A_1 & \xrightarrow{f_1} & B_1 & \xrightarrow{f_2} & C_1 & \xrightarrow{f_3} & D_1 & \xrightarrow{f_4} & E_1 & \xrightarrow{f_5} & \{0_{E_1}\} \\
 & & \alpha_1 \downarrow\cong & & \alpha_2 \downarrow\cong & & \alpha_3 \downarrow & & \alpha_4 \downarrow\cong & & \alpha_5 \downarrow\cong & & \\
\{0_{A_2}\} & \xrightarrow{g_0} & A_2 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & C_2 & \xrightarrow{g_3} & D_2 & \xrightarrow{g_4} & E_2 & \xrightarrow{g_5} & \{0_{E_1}\}
\end{array}$$

Show that if α_1, α_2, α_4, α_5 are isomorphisms, then α_3 is an isomorphism.

試題隨卷繳回

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$$\begin{aligned} V^2 = &\|\vec{\mathbf{u}}\|^2 \, \|\vec{\mathbf{v}}\|^2 \, \|\vec{\mathbf{w}}\|^2 \; + \; 2\, (\vec{\mathbf{u}} \cdot \vec{\mathbf{v}})\, (\vec{\mathbf{v}} \cdot \vec{\mathbf{w}})\, (\vec{\mathbf{w}} \cdot \vec{\mathbf{u}}) \\ &- \|\vec{\mathbf{u}}\|^2\, (\vec{\mathbf{v}} \cdot \vec{\mathbf{w}})^2 - \|\vec{\mathbf{v}}\|^2\, (\vec{\mathbf{w}} \cdot \vec{\mathbf{u}})^2 - \|\vec{\mathbf{w}}\|^2\, (\vec{\mathbf{u}} \cdot \vec{\mathbf{v}})^2 \end{aligned}$$

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