國立臺灣師範大學 105 學年度碩士班招生考試試題

科目：**基礎數學**　　　　適用系所：**數學系**

注意：1.本試題共 2 頁，請依序在答案卷上作答，並標明題號，不必抄題。2.答案必須寫在指定作答區內，否則依規定扣分。

Part I : Calculus

1. (28 points) Evaluate the integrals

   (a) $\int \cos\sqrt{x}dx$

   (b) $\int xe^{x^2}dx$

   (c) $\int \frac{1}{(x-1)x^2}dx$

   (d) $\int_1^{25} \frac{\log_5 x}{x}dx$

2. (3 points) Let $\int_{x^2}^{2} f(t^3)dt = \frac{1}{2}\cos(\frac{\pi}{2}x)$, where f is a continuous function. Compute $f(1)$.

3. (3 points) Find a power series for $\sin(x^2)$, centered at 0.

4. (4 points) Find a real number a such that the following limit is correct.

$$\lim_{x\to 0}\left[(x^2-x)\left(\cos\left(\frac{1}{x}\right)+1\right)^2+a\right]=1$$

5. (4 points) Let $f(x)=5x(1-5x^2)$ and $g=\underbrace{f\circ\cdots\circ f}_{100}$. Find $g^{'}(\sqrt{6}/5)$.

   Note that $\underbrace{f\circ\cdots\circ f}_{3}$ denotes $f(f(f(x)))$.

6. (3 points) Let

$$\int_0^1\int_{1-y}^{\sqrt{1-y^2}} f(x,y)dxdy=\int_a^b\int_c^d f(x,y)dydx,$$

   where f be a continuous function. Find a, b, c and d.

7. (5 points) Evaluate the integral

$$\int_2^4\int_{2/y}^{y/2}\sqrt{\frac{y}{x}}e^{\sqrt{xy}}dxdy.$$

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Part II : Linear algebra

8. (10 points) Given 5 points $(a_1, b_1), (a_2, b_2), (a_3, b_3), (a_4, b_4), (a_5, b_5)$ in the plane R^2, where a_1, a_2, a_3, a_4, a_5 are distinct. Use matrix methods to show that there is a polynomial $f(x)$ of degree at most 4, such that the above 5 points lie on the graph of $y = f(x)$.

9. Let P_2 be the vector space of all polynomials of degree at most 2, and $T : P_2 \longrightarrow P_2$ be the linear transformation defined by $T(f(x)) = 3f'(x) + f(x)$ for all $f(x) \in P_2$. Given two ordered bases $B_1 = \{1, x, x^2\}$ and $B_2 = \{1, x, x + x^2\}$ of the vector space P_2 .

(a) (4 points) Find the matrix representation A of T relative to B_1 and B_2 .

(b) (3 points) Use the matrix A to compute $T(2x^2 - x + 1)$.

(c) (3 points) Find the general formula of $T^{-1}(ax^2 + bx + c)$.

10. (10 points) A hyperplane in R^n is a subspace of dimension $n - 1$. Prove that S is a hyperplane in R^n if, and only if, there is a vector $a \in R^n$ such that $S = \{x \in R^n \mid \langle x, a \rangle = 0\}$, where $\langle \cdot, \cdot \rangle$ denotes the usual inner product on R^n.

11. Let $A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$.

(a) (5 points) Find an orthogonal matrix C such that $C^{-1}AC = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

(b) (5 points) Find an invertible matrix P such that $P^3 = A$.

12. (10 points) Let V and W be two vector spaces having the same finite dimension. Prove that a linear transformation $T : V \longrightarrow W$ is one-to-one if , and only if, T is onto.

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