國立政治大學 104 學年度研究所 碩士班招生考試試題

考 試 科 目	統計學B 41221	所 別	金融學系	考 試 時 間	2 月 28 日(六)第二節

1. Assume that T is a continuous random variable with probability density function (p.d.f.) $f(t)$ and cumulative distribution function (c.d.f.) $F(t) = \Pr(T < t)$, which gives the probability that the event has occurred by duration t. It will often be convenient to work with the complement of the c.d.f, the survival function

$$S(t) = \Pr\{T \geq t\} = 1 - F(t) = \int_t^\infty f(x)dx \tag{1.1}$$

which gives the probability of being alive just before duration t, or more generally, the probability that the event of interest has not occurred by duration t.

The distribution of T can also be given by the hazard function, or instantaneous rate of occurrence of the event, defined as

$$\lambda(t) = \lim_{dt \to 0} \frac{\Pr\{t \leq T < t + dt \mid T \geq t\}}{dt}. \tag{1.2}$$

The numerator of this expression is the conditional probability that the event will occur in the interval $[t, t+dt)$ given that it has not occurred before, and the denominator is the width of the interval. Dividing one by the other we obtain a rate of event occurrence per unit of time. Taking the limit as the width of the interval goes down to zero, we obtain an instantaneous rate of occurrence.

The conditional probability in the numerator may be written as the ratio of the joint probability that T is in the interval $[t, t+dt)$ and $T \geq t$ (which is, of course, the same as the probability that t is in the interval), to the probability of the condition $T \geq t$. The former may be written as $f(t)dt$ for small dt, while the latter is $S(t)$ by definition. Dividing by dt and passing to the limit gives the useful result

$$\lambda(t) = \frac{f(t)}{S(t)}. \tag{1.3}$$

which is another definition hazard function. More explicitly, the rate of occurrence of the event at duration t equals the density of events at t, divided by the probability of surviving to that duration without

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experiencing the event.

Note from Equation (1.1) that $-f(t)$ is the derivative of $S(t)$. This suggests rewriting Equation (1.3) as

$$\lambda(t) = -\frac{d}{dt}\log S(t). \tag{1.4}$$

(a) If we now integrate from 0 to t and introduce the boundary condition $S(0)=1$ (since the event is sure not to have occurred by duration 0), please find the survival function $S(t)$. (5%)

This result shows that the survival and hazard functions provide equivalent characterizations of the distribution of T. Given the survival function, we can always differentiate to obtain the density and then calculate the hazard using Equation (1.4). Given the hazard, we can always integrate to obtain the cumulative hazard and then exponentiate to obtain the survival function using the answer of problem (a). For example, the simplest survival distribution is obtained by assuming a constant risk over time, so the hazard is $\lambda(t)=\lambda$ for all t.

(b) Please find the survival function with the previous example. (5%)

(c) Please find the probability density function and the mean of T with the previous example. (10%)

2. Consider a Brownian motion process $\{B(t), t \geq 0\}$. $B(0)=0$ and $B(t)$ is normal with mean 0 and variance t, where its density function is given by

$$f_t(b) = \frac{1}{\sqrt{2\pi t}} e^{-b^2/2t},$$

and the process $B(t)$ has stationary and independent increments, where $B(t_1)$, $B(t_2)-B(t_1)$, …, $B(t_n)-B(t_{n-1})$ for $t_1 < \ldots < t_n$ are independent and $B(t_k)-B(t_{k-1})$ is normal with mean 0 and variance $t_k - t_{k-1}$. Let the dynamics of the stock price be $S(T)=S(0)\exp\left\{(r-\frac{1}{2}\sigma^2)T+\sigma B(T)\right\}$ under the risk neutral measure at time $t_n = T$.

(a). Please find the covariance of $B(t)$ and $B(s)$, $Cov(B(t), B(s))$. (5%)

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(b). Please find the mean of the discounted stock price, $\exp\{-rT\}\,S(T)$, with the risk-neutral measure at the time T. ($E(\exp\{-rT\}S(T))$). (10%)

(c). Please compute $E(\exp\{-rT\}K1_{\{S(T)>K\}})$ and $E(\exp\{-rT\}S(T)1_{\{S(T)>K\}})$ under the risk neutral measure where $1_{\{\cdot\}}$ denotes the indicator function and K is a constant number. (20%)

(d). How to estimate the volatility of the stock (σ) when you have data of the stock price every year? (5%)

3. A stochastic process $\{X(t), t \geq 0\}$ is said to be a compound Poisson process if it can be represented as

$$X(t) = \sum_{i=1}^{N(t)} Y_i, \qquad t \geq 0$$

where $\{N(t), t \geq 0\}$ is a Poisson process, and $\{Y_i, i \geq t\}$ is a family of independent and identically distributed random variables which are also independent of $\{N(t), t \geq 0\}$. Suppose that the only 'abnormal' vibrations in stock price due to the arrival of important new information about the stock and the important information arrives only at discrete points in time. The new information follows a Poisson process with a arrival rate λ. If Y_i denotes the (down or up) size of the stock return when the new information happens, $i = 1, 2, \ldots$, are independent and identically distributed from normal density with mean μ and variance σ^2. Assume that the stock price at the time T is $S(T) = S(0)\exp\{X(t)\} = S(0)\exp\left\{\sum_{i=1}^{N(t)} Y_i\right\}$. (This component is modeled by a 'jump' process, $\exp\{X(t)\}$, reflecting the impact of the information.)

(a). Please find the mean of $S(T)$ at the time T, $E(S(T))$. (10%)

(b). Please compute $E(1_{\{S(T)>K\}})$ and $E(S(T)1_{\{S(T)>K\}})$ where $1_{\{\cdot\}}$ denotes the indicator function and K is a constant number. (20%)

(c). How to estimate λ, μ and σ^2 when you have data of the stock price every year? (10%)

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