考試科目 Course	線性代數	系級	應用數學	日期 Date, Period	4月22日 第3:20-5:00節	試題編號 Course No.

說明：

一.總共四大題，每大題含十小題（分別依編號[1],[2],[3],...,[10]標示於題句中），每小題計3分（亦即每大題為30分），共120分。

二.作答時，大題中之小題不可顛倒順序。凡答案錯誤，該小題即不予計分。得分如超過100分，則仍以100分計。

三.在答案卷上，請清楚標明題號及簡潔的答案（演算或證明步驟不必列出），如下例：

1. [1] True
   [2] False
   [3] $A=\begin{bmatrix}1&2\\3&4\end{bmatrix}$
   $\vdots$
   [10] $T(M)=\{M \mid M=\begin{bmatrix}a&b\\c&d\end{bmatrix}, a,b,c,d\in\mathbb{R}\}$

2. [1] $L=\begin{bmatrix}1&0&0\\2&1&0\\3&2&1\end{bmatrix}$
   [2] $\beta=100$
   [3] Eigenvalues are 1, 2, 3, 4.
   [4] $x=\begin{bmatrix}1\\2\\3\end{bmatrix}$
   $\vdots$
   [10] $A^{-1}=\begin{bmatrix}4&3\\2&1\end{bmatrix}$

1. (a) Let $T: \mathrm{V}\to\mathrm{W}$ be a linear transformation, where V, W are vector spaces with bases $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$, $\{\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3\}$ respectively. Suppose $T(\mathbf{v}_1)=\mathbf{w}_1+\mathbf{w}_2+\mathbf{w}_3$, $T(\mathbf{v}_2)=\mathbf{w}_2+\mathbf{w}_3$, and $T(\mathbf{v}_3)=\mathbf{w}_3$. Find the matrix $\mathbf{A}$ [1] for T using these basis vectors. What input vector $\mathbf{v}$ [2] gives $T(\mathbf{v})=\mathbf{w}_1$? What are $T^{-1}(\mathbf{w}_1)$ [3], $T^{-1}(\mathbf{w}_2)$ [4], and $T^{-1}(\mathbf{w}_3)$ [5]? Find all $\mathbf{v}$'s [6] that satisfy $T(\mathbf{v})=\mathbf{0}$. (b) Suppose $T:\mathrm{V}\to\mathrm{V}$ with $T(\mathbf{M})=\begin{bmatrix}1&0\\0&0\end{bmatrix}\mathbf{M}\begin{bmatrix}0&0\\0&1\end{bmatrix}$, where V contains all 2-by-2 matrices $\mathbf{M}$. Find a matrix $\mathbf{M}$ [7] with $T(\mathbf{M})\neq\mathbf{O}$ (the zero matrix). Describe all matrices $\mathbf{M}$ [8] with $T(\mathbf{M})=\mathbf{O}$ (the kernel of T) and all output matrices $T(\mathbf{M})$ [9] (the range of T). Suppose $S:\mathrm{V}\to\mathrm{V}$ with $S(\mathbf{M})=\mathbf{AMB}$, where $\mathbf{A}$ and $\mathbf{B}$ are invertible 2-by-2 matrices. Then $S^{-1}(\mathbf{M})=$ [10].

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2-2

2. Let $\mathbf{L}$ be a *unit* lower triangular matrix (the diagonal elements of $\mathbf{L}$ are all ones) and $\mathbf{U}$ an upper triangular matrix. (a) Assume $\mathbf{A} = \mathbf{LU}$ and $\mathbf{B} = \mathbf{UL}$. Determine the following statements as being true or false: (i) $\mathbf{A}$ and $\mathbf{U}$ have the same column space [1]. (ii) $\mathbf{A}$ and $\mathbf{U}$ have the same nullspace [2]. (iii) $\det(\mathbf{A}) = \det(\mathbf{U})$ [3]. Express $\mathbf{B}$ [4] in terms of $\mathbf{A}$ and $\mathbf{L}$ by eliminating $\mathbf{U}$. If $\mathbf{A}$ is singular, so is $\mathbf{B}$, true or false [5]? What is the relation [6] between the eigenvalues of $\mathbf{A}$ and those of $\mathbf{B}$? (b) Find the solution of $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 11 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 8 \\ 10 \\ 14 \end{bmatrix}$, by the following procedure: (i) First factor $\mathbf{A}$ into the product of $\mathbf{L}$ [7] and $\mathbf{U}$ [8]. (ii) Then solve $\mathbf{Ly} = \mathbf{b}$ for $\mathbf{y}$ [9] by forward substitution. (iii) Finally solve $\mathbf{Ux} = \mathbf{y}$ to get $\mathbf{x}$ [10] by back substitution.

3. Consider the real n-by-n matrix $\mathbf{A} = \mathbf{I} + 2\mathbf{u}\mathbf{u}^{\mathsf{T}}$, where $\mathbf{u} \in \mathbb{R}^n$ (a column vector with n real components) and $\|\mathbf{u}\|_2 = \sqrt{\mathbf{u}^{\mathsf{T}}\mathbf{u}} = 1$ ($\mathbf{u}^{\mathsf{T}}$, a row vector, is the transpose of $\mathbf{u}$). (a) Compute $\mathbf{Au}$ and find an eigenvalue [1] of $\mathbf{A}$. For this fixed $\mathbf{u}$, how many [2] linearly independent vectors (up to a scalar multiple) $\mathbf{x} \in \mathbf{R}^n$ can you find such that $\mathbf{u}^{\mathsf{T}}\mathbf{x} = 0$? Now, by choosing such $\mathbf{x}$ and computing $\mathbf{Ax}$, determine all the n eigenvalues [3] of $\mathbf{A}$. (b) Let $\mathbf{E} = \mathbf{I} - \sigma\mathbf{u}\mathbf{v}^T$, $\mathbf{F} = \mathbf{I} - \tau\mathbf{u}\mathbf{v}^T$, where σ, τ are real and *nonzero*, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ with $\|\mathbf{u}\|_2 = \|\mathbf{v}\|_2 = 1$. Then $\mathbf{EF} = \mathbf{I}$ if $\mathbf{v}^T\mathbf{u} =$ [4]; if $\sigma\mathbf{v}^T\mathbf{u} =$ [5], then $\mathbf{E}$ is singular. Can $\mathbf{A}$ be singular for some $\mathbf{u}$ [6]? Why? Find $\mathbf{A}^{-1}$ [7] if $\mathbf{A}$ is invertible. What can you say about the Jordan form of $\mathbf{A}$ [8]? (c) Let $\mathbf{G} = \mathbf{I} + \mathbf{u}\mathbf{v}^{\mathsf{T}}$, where $\mathbf{u}, \mathbf{v}$ are defined as in part (b). Give the Jordan form of $\mathbf{G}$ (i.e., describe $\mathbf{J}$ such that $\mathbf{G} = \mathbf{XJX}^{-1}$ for some nonsingular $\mathbf{X}$) in two different cases: (i) $\mathbf{v}^{\mathsf{T}}\mathbf{u} \neq 0$ [9] (ii) $\mathbf{v}^{\mathsf{T}}\mathbf{u} = 0$ [10].

4. (a) Matrices $\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$ and $\begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix}$ are similar, true or false [1]? (b) Which of the following matrices is/are not diagonalizable [2]? (i) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ (iii) $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ (iv) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (v) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (vi) $\begin{bmatrix} i & i \\ i & i \end{bmatrix}$, $i = \sqrt{-1}$ (vii) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. (c) Let $\mathbf{A} = \begin{bmatrix} 2 & \beta \\ 1 & 0 \end{bmatrix}$. If $\beta \neq$ [3], then $\mathbf{A}$ is diagonalizable; if $\beta =$ [4], then $\mathbf{A}$ is orthogonally diagonalizable (i.e., $\mathbf{A} = \mathbf{Q\Lambda Q}^{\mathsf{T}}$, where $\mathbf{Q}$ is orthogonal and $\mathbf{\Lambda}$ is diagonal). (d) Let $\mathbf{P}$ ($\neq \mathbf{O}$) be an orthogonal projection matrix (i.e., $\mathbf{P}^2 = \mathbf{P}$ and $\mathbf{P}^{\mathsf{T}} = \mathbf{P}$) from $\mathbb{R}^n$ onto a subspace of dimension $k < n$. What is the minimal polynomial [5] for $\mathbf{P}$? What are the n eigenvalues [6] of $\mathbf{P}$? (e) Find the four eigenvalues of: (i) $\mathbf{A} = \begin{bmatrix} 7 & 1 & 1 & 1 \\ 1 & 7 & 1 & 1 \\ 1 & 1 & 7 & 1 \\ 1 & 1 & 1 & 7 \end{bmatrix}$ [7] (Hint: Consider the rank of $\mathbf{A} - \lambda\mathbf{I}$ for some λ.) (ii) $\mathbf{B} = \frac{1}{\sqrt{3}} \begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{bmatrix}$ [8] (Hint: $\mathbf{B}$ is skew-symmetric and orthogonal.) (f) Determine the 2-by-2 matrix $\mathbf{A}$ [9] such that $f(x_1, x_2) =$