科目	數學	所別	經濟甲組	考試時間	4月18日(上/下)午第2節 星期日

%) **1.** Consider the matrix $\mathbf{A}$ and the vector $\mathbf{v}$,

$$\mathbf{A} = \begin{pmatrix} -1 & 1 & 2 \\ -6 & 2 & 6 \\ 0 & 1 & 1 \end{pmatrix} \qquad \mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(5%) **(a)** Show that $\mathbf{v}$ is an eigenvector of $\mathbf{A}$ and find the corresponding eigenvalue.

(b) Diagonalize $\mathbf{A}$ by finding the eigenvalues and corresponding eigenvectors. Write down an invertible matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$.

Check that $\mathbf{AP} = \mathbf{PD}$.

(10%) Find $\mathbf{P}^{-1}$.

(c) The system of linear difference equations $\mathbf{x}_t = \mathbf{A}\mathbf{x}_{t-1}$ for $t \in \mathbb{N}$ is given by

$$\begin{aligned} x_t &= -x_{t-1} + y_{t-1} + 2z_{t-1} \\ y_t &= -6x_{t-1} + 2y_{t-1} + 6z_{t-1} \\ z_t &= y_{t-1} + z_{t-1} \end{aligned} \qquad \text{with} \qquad \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

(20%) Solve the system. (Show that $\mathbf{x}_t = \mathbf{A}^t\mathbf{x}_0$ and express $\mathbf{A}^t$ in terms of $\mathbf{P}$, $\mathbf{P}^{-1}$ and $\mathbf{D}$.)

Write down expressions for x_t, y_t, z_t.

Find the value of z_5 from your solution.

%) **2.** Consider the system of linear equations $\mathbf{Ax} = \mathbf{b}$, where λ and μ are constants:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 5 & 1 & \lambda \\ 1 & -1 & 1 \end{pmatrix} \qquad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 2 \\ 7 \\ \mu \end{pmatrix}$$

(5%) (i) Calculate the determinant of $\mathbf{A}$, $|\mathbf{A}|$

(ii) Determine for which values of λ and μ this system has:

(20%)
- (a) a unique solution
- (b) no solutions
- (c) infinitely many solutions.

In case (a), use Cramer's rule to find the value of z in terms of λ and μ.

In case (c), solve the system using Gaussian elimination and express the solution in vector form, $\mathbf{x} = \mathbf{p} + t\mathbf{v}$.

0%) 3. (a) Find and classify the stationary points of the function: $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = 4x^2y - 9y - 7x^2$$

(10%) Does f have any global extrema?

(b) (i) Find the overall minimum for the joint cost function

$$C(x, y) = 3x^2 + 2xy + 3y^2 - 32x - 32y + 163$$

for a company producing goods x and y.
The production function for the company for the goods x and y is

$$P(x, y) = 100x^{\frac{1}{4}}y^{\frac{1}{4}}.$$

Evaluate the production when cost is minimised.

(10%) (ii) Show that when the production must be at least 300, the cost of production is minimised when $(x, y) = (9, 9)$.

(10%) (iii) Given that the production must be at least 100, find the quantities for which the cost of production is minimised.

(c) (i) Sketch a clear map of the contours of the surface

$$z = 3x^2 - 10xy + 3y^2$$

in the xy plane, roughly indicating the directions of their gradient vectors.

(10%) (ii) Find, if they exist, the points where the function $f(x, y) = 3x^2 - 10xy + 3y^2$ attains its maximum and minimum values subject to the constraints

$$x + y \leq 5, \quad y \geq 0$$