1. Multiple Choice (12%)
Instructions:

- There are 4 questions, each of which is associated with 5 possible responses.
- For each of questions, select **ONE most appropriate** response.
- For each response you provide, **you will be awarded 3 marks if the response is correct and -3 marks if the response is incorrect (答錯一題倒扣三分).**
- You get 0 mark if no response is provided.

(1.1) Let $\boldsymbol{L}[\cdot]$ denotes the Laplace transform.
(A) The Laplace transform is a linear operation.
(B) If $\boldsymbol{L}[\mathrm{f(t)}] = \mathrm{F(s)}$, then $\boldsymbol{L}[\mathrm{t^2 f(t)}] = \frac{d^2}{ds^2}\mathrm{F(s)}$.
(C) If $\boldsymbol{L}[\mathrm{f(t)}] = \mathrm{F(s)}$, then $\boldsymbol{L}\left[\frac{d}{dt}\mathrm{f(t)}\right] = \mathrm{sF(s)} - \mathrm{f(0)}$.
(D) If $\boldsymbol{L}[\mathrm{f(t)}] = \mathrm{F(s)}$ and $\boldsymbol{L}[\mathrm{g(t)}] = \mathrm{G(s)}$, then $\boldsymbol{L}[\mathrm{f} * \mathrm{g(t)}] = \mathrm{F(s)G(s)}$, where $*$ denotes the convolution integral.
(E) All of the above statements are TRUE.

(1.2) For what ω does the sinusoidal solution of $\ddot{x} + \dot{x} + x = \cos(\omega t)$ have the biggest amplitude?
(A) 2 (B) 1 (C) $\sqrt{2}$ (D) $1/\sqrt{2}$ (E) $1/2$.

(1.3) Consider the ODE $\ddot{x} + x\dot{x}^2 + (x^2 - 1) = 0$, where x is a real function.
(A) This is a time-invariant ODE.
(B) This ODE is nonlinear.
(C) This ODE has two equilibria.
(D) The equilibria of this ODE are -1 and 1.
(E) All of the above are TRUE.

(1.4) For what value of (a, b) will the solutions to $\ddot{y} + a\dot{y} + by = 0$ exhibit oscillatory behavior ?
(A) (1,2) (B) (1,0) (C) $(1, -1)$ (D) (2,1) (E) All of the above

2. (13%) Consider the following system of differential equations:

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -2x_1(t) - 3x_2(t) + u(t)$$
$$y(t) = 2x_1(t) - x_2(t)$$

with initial conditions $x_1(0) = x_2(0) = 0$. The external forcing function u(t) is as follows

$$u(t) = \begin{cases} 1 & \text{if } t \geq 1 \\ 0 & \text{if } t < 1 \end{cases}$$

See Figure 1 for an illustration.

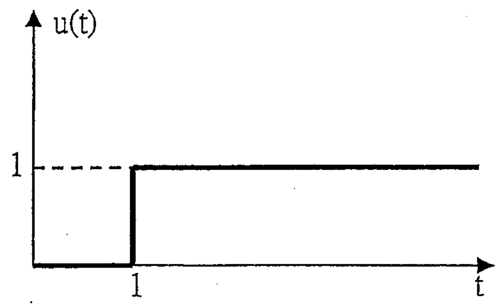

Figure 1: illustration of u(t).

(2.1) (10%) Find the corresponding response y(t).

(2.2) (3%) Calculate the peak value and the steady state value of y(t).

3. (20%) Let $\{\mathbf{x},\mathbf{y},\mathbf{z}\}$ be a set of linearly independent vectors in $\mathbb{R}^n$, and let $S := Span(\mathbf{x},\mathbf{y})$ and $T := Span(\mathbf{y},\mathbf{z})$. Define matrix $A := \mathbf{x}\mathbf{y}^T + \mathbf{y}\mathbf{z}^T$. Obviously, the sets S, T, their orthogonal complements $S^{\perp}$, $T^{\perp}$, and the four sets associated with matrix A, i.e. the two ranges $R(A)$ and $R(A^T)$ and the two null spaces $N(A)$ and $N(A^T)$, are all subspaces of $\mathbb{R}^n$.

This problem has three questions. The first one is a MULTIPLE-choice question, for which you don't need to give any derivation, but **you need to give detailed derivations for the other two questions**. In the multiple-choice question, the total score is evenly divided into each correct statement, and your each correct choice will get the partial score. However, the penalty for each wrong choice is equal to the score of each correct choice. (所以同時選了一個對的答案和一個錯的答案時，淨得分為 0；但是扣分僅扣到該小題 0 分為止。另外為方便改題、請將選擇題的答案寫在此題做答處即可，不要寫到別處，以免漏改。)

(3.1) What are the possible relationships associated with S and $S^{\perp}$? (6%)

(A) $S^{\perp} \subset N(A^T)$

(B) $N(A^T) \subset S^{\perp}$

(C) $S^{\perp} \subset N(A)$

(D) $S \subset R(A)$

(E) $R(A^T) \subset S$

(3.2) Similar to the sub-question (3.1), please find out all possible relationships of T associated with the subspaces in the set $\{R(A), R(A^T), N(A), N(A^T)\}$. Give detailed arguments for your answers. (6%)

(3.3) Now let $(\lambda, \mathbf{v})$ be an eigenpair of matrix A with $\lambda \neq 0$. Then from the definition of A, it can be shown that $\mathbf{v}$ lies in certain subspace of $\mathbb{R}^n$ and λ is an eigenvalue of another matrix, denoted by $B \in \mathbb{R}^{m\times m}$ with $m = rank(A)$. Please (i) (2%)

indicate the subspace of $\mathbb{R}^n$ where the eigenvector $\mathbf{v}$ lies, and (ii)(6%) use vectors $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ to describe the matrix B.

4. (25%) Let P_2 denote the vector space of all polynomials of degree less than 2. Consider the transformation $L: P_2 \to \mathbb{R}^2$ defined by $L(p(x)) := \begin{bmatrix} \int_0^{\alpha} p(x) \\ p(\beta) \end{bmatrix}$ with **undecided parameters** $\alpha > 0$ and $\beta \in \mathbb{R}$. Let A be the matrix representation of transformation L with respect to the ordered bases $E = [1, x]$ and $E' = \left[\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right]$ for P_2 and $\mathbb{R}^2$, respectively.

以下小題僅需依序寫下答案即可，不需做任何推導。

(4.1) Find the set of $\begin{bmatrix}\alpha\\ \beta\end{bmatrix} \in \mathbb{R}^2$ such that matrix A becomes singular. (5%)

(4.2) Let's define an inner product for P_2 by $\langle p(x), q(x) \rangle := \sum_{i=1}^{2} p(x_i)q(x_i)$, for arbitrary $p(x), q(x) \in P_2$, with $\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}\gamma\\1\end{bmatrix}$ and $\gamma \neq 1$ an **undecided parameter.** Find the orthonormal basis, denoted by $F := [\mathbf{f}_1, \mathbf{f}_2]$, of P_2 generated from basis E given above to satisfy the subspace equality constraints $Span(\mathbf{f}_1) = Span(1)$ and $Span(\mathbf{f}_1, \mathbf{f}_2) = Span(1, x)$. (8%)

(4.3) Let B denote the matrix representation of transformation L with respect to the ordered bases F computed in (4.2) and $F' = \left[\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right]$ for P_2 and $\mathbb{R}^2$, respectively. Find the matrix B. (6%)

(4.4) Now suppose $\alpha = \sqrt{2}$ and $\beta = 0$. Find all possible values of γ such that the set of eigenvalues of B is $\{1, \sqrt{2}\}$. (6%)

5.(a)(7%)Let $f(z)$ be a complex function defined by

$$f(z) = \begin{cases} \bar{z}^2 / z, & \text{if } z \neq 0 \\ 0, & \text{if } z = 0 \end{cases},$$

where $\bar{z}$ denotes the complex conjugate of the complex variable z. Does the function $f(z)$ satisfy the Cauchy-Riemann equations? Give your reason (no credit will be given if there is no explanation).

(b)(8%) Does the derivative of $f(z)$ at $z = 0$, i.e., $f'(0)$, exist? Give your reason (no credit will be given if there is no explanation).

6. (15%) Using the theory of Residues, compute the inverse $f(t)$, $-\infty < t < \infty$, of the Fourier transform

$$F(\omega) = \frac{2a}{a^2 + \omega^2}, \quad a > 0.$$