考 試 科 目	統計學 A	所 別	金融學系 4121	考 試 時 間	2 月 22 日(六)第三節

1. Let X_t be a normally distributed random variable with mean rt and variance $\sigma^2 t$, i.e. $X_t \sim N(rt, \sigma^2 t)$, for a stock return and a time period [0,*t*]. If $S_t = S_0 e^{X_t}$ where S_t is the stock price at time *t*, then the stock price S_t is said to be lognormally distributed random variable.
   (1). (5%) Find the probability $P(S_t > K)$ for a positive real number K and a given cumulative standard normal distribution $N(\cdot)$.(i.e., the probability which the stock price exceeds K at time t).
   (2). (10%) Compute $E\left((S_t - K)\ 1_{\{S_t > K\}}\right)$, where $1_{\{S_t>K\}}$ denotes the indicator function. (i.e, the expected profit $(S_t - K)$ when the stock price exceeds K at time t)

2. Suppose that the joint distribution of the random variable's X and Y is the Bivariate Normal distribution. That is

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-q/2}$$

where $q = \frac{1}{1-\rho^2}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]$, $-\infty < x < \infty$, and $-\infty < y < \infty$

   (1). (5%) Find the marginal distribution of X and Y.
   (2). (5%) Derive the conditional distribution, the conditional mean, and the conditional variance of Y given $X = x$.
   (3). (10%) Find the MLE (maximum likelihood estimator) of μ_X, μ_Y, σ_X^2, σ_Y^2, and ρ.
   (4). (5%) Prove the MLE of σ_X^2 is asymptotic unbiased.

3. Consider the regression model as $Y_t = \alpha + \beta X_t + \varepsilon_t$, assume ε_t, $t = 1,..,n$, are independent, $E(\varepsilon_t) = 0$ for all t, and $Var(\varepsilon_t) = \sigma^2$ for all t.
   (1). (5%) Find the least squares estimates $\hat{\alpha}$ and $\hat{\beta}$, and check that the estimators are unbiased.
   (2). (10%) Describe the Gauss-Markov Theorem, and prove the Gauss-Markov theorem for the least squares estimator $\hat{\beta}$ of β.

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(3). (10%) If we have a linear regression model with heteroskedasticity (the variance assumption of the residuals change from $Var(\varepsilon_i)=\sigma^2$ to $Var(\varepsilon_i)=\sigma_i^2$), then whether the least squares estimator $\hat{\beta}$ is still a linear and unbiased estimator or not? And find the variance of the least square estimator $\hat{\beta}$.

(4). (5%) If the true model is $Y_i=\beta X_i+\varepsilon_i$, find the least squares estimator $\tilde{\beta}$ of β, and prove $\tilde{\beta}$ is unbiased?

(5). (10%) If the true model has a constant term so that $Y_i=\alpha+\beta X_i+u_i$, show that $\tilde{\beta}$ is biased, which you use a wrong model (the regression model for no constant term) to estimate. Please derive the condition under which $\tilde{\beta}$ will be unbiased even though the wrong model was used.

4. Assume that $X_{ij}\sim N(\mu_{ij},\sigma^2)$, $i=1,2,\ldots,a$, and $j=1,2,\ldots,b$; and the $n=ab$ random variables are independent. Assume that the mean μ_{ij} are composed of a row effect with a levels (factor A), a column effect with b levels (factor B), and an overall effect in some additive way, namely $\mu_{ij}=\mu+\alpha_i+\beta_j$, where $\sum_{i=1}^{a}\alpha_i=0$ and $\sum_{j=1}^{b}\beta_j=0$. The parameter α_i represents the i th row effect, and the parameter β_j represents the jth column effect. Let

$$\bar{X}_{i.}=\frac{1}{b}\sum_{j=1}^{b}X_{ij},\quad \bar{X}_{.j}=\frac{1}{a}\sum_{i=1}^{a}X_{ij},\quad \bar{X}_{..}=\frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b}X_{ij},$$

$$\mathrm{SS(TO)}=\sum_{i=1}^{a}\sum_{j=1}^{b}\left(X_{ij}-\bar{X}_{..}\right)^2,\quad \mathrm{SS(A)}=b\sum_{i=1}^{a}\left(\bar{X}_{i.}-\bar{X}_{..}\right)^2,$$

$$\mathrm{SS(B)}=a\sum_{j=1}^{b}\left(\bar{X}_{.j}-\bar{X}_{..}\right)^2,\quad \mathrm{SS(E)}=\sum_{i=1}^{a}\sum_{j=1}^{b}\left(X_{ij}-\bar{X}_{i.}-\bar{X}_{.j}+\bar{X}_{..}\right)^2$$

where SS(TO) denotes the total sum of squares, SS(A) is the sum of squares among levels of factor A, SS(B) presents the sum of squares among levels of factor B, and SS(E) is the residual sum of squares.

(1). (10%) Prove SS(TO)=SS(A)+SS(B)+SS(E) in detail.

(2). (10%) Derive the statistic for testing the null hypothesis Ha: $\alpha_1=\alpha_2=\ldots=\alpha_a=0$ against all alternatives, and find the rejection region.

備註	試題隨卷繳交