國立中山大學102學年度碩士暨碩士專班招生考試試題

科目名稱：工程數學甲【電機系碩士班甲組、丙組選考、丁組、戊組、己組】　題號：431002

※本科目依簡章規定「可以」使用計算機（廠牌、功能不拘）　

1. (7%) Find the Laurent series representation of a function

$$f(z)=\frac{5z+j2}{z^2+jz}$$

with center at $z=j$ in the domain $1<|z-j|<2$, $j=\sqrt{-1}$.

2. (8%) Evaluate the following integral:

$$\int_C \frac{z^3 e^{1/z}}{1+z^3}dz,$$

where C denotes a counterclockwise simple closed contour $|z|=3$.

3. (15%) Compute the Fourier transform $F(\omega)=\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$ of a signum function $f(t)$ defined as

$$f(t)=\begin{cases}-1, & t<0\\ 0, & t=0\\ 1, & t>0\end{cases}$$

Each calculation step is required for obtaining the credit.

4. (15%) 下面的問題共有三個子題，**只要寫出每個子題的答案即可**(不需寫出計算過程)，例如：(a) $\beta=1,\gamma=2$。
Let $\mathbf{a}_1=\begin{bmatrix}1 & 0 & 1 & \alpha\end{bmatrix}^T$, $\mathbf{a}_2=\begin{bmatrix}1 & \beta & 2 & 2\end{bmatrix}^T$, and $\mathbf{a}_3=\begin{bmatrix}-2 & 3 & \gamma & -4\end{bmatrix}^T$ be three vectors in $\mathbb{R}^4$, where α, β, and γ are three real parameters, and denote $A:=\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3\end{bmatrix}\in\mathbb{R}^{4\times 3}$.
   (a) (4%) Suppose α is not a positive integer. Find real β and γ such that $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ is a linearly dependent set.
   (b) (5%) Now let $\alpha=2, \beta=-1, \gamma=-5$, and let $\mathbf{x}$ be a nonzero vector in the null space $N(A)$ of A. Find the value of k to satisfy $\|\mathbf{x}\|_1+2\|\mathbf{x}\|_\infty+k\|\mathbf{x}\|_2=0$.
   (c) (6%) Now let $\alpha=2, \beta=-1, \gamma=-5$, and let d denote the distance between vector $\begin{bmatrix}1 & 4 & 0\end{bmatrix}^T$ and $R(A^T)$, the range space of A^T. Compute the value of d.

5. (10%) 下面的問題共有二個子題，**只要寫出每個子題的答案即可**(不需寫出計算過程)，例如：(a) $\theta=30°$或$\theta=\pi/6$。
Consider the inner product space $C[0,1]$ with $\langle f,g\rangle:=\int_0^1 f(x)g(x)dx$ and the norm $\|f\|:=\sqrt{\langle f,f\rangle}$.
Denote $S:=\text{span}\{1,x\}$ as a subspace of $C[0,1]$.
   (a) (4%) Compute the angle θ, taken value in $[0,\pi/2]$, between 1 and x.
   (b) (6%) Find a vector $u(x)$ in $C[0,1]$, so that $\{1,u(x)\}$ forms an orthonormal basis for S.

6. (20%) Consider the following system of differential equations:

$$\dot{x}_1(t) = a_{11}x_1(t) + a_{12}x_2(t)$$
$$\dot{x}_2(t) = a_{21}x_1(t) + a_{22}x_2(t) + u(t)$$

where $a_{11}, a_{12}, a_{21}, a_{22}$ are constant coefficients.

(a) (5%) Suppose $u \equiv 0$ and the equations are driven by non-zero initial conditions. Determine the conditions on the coefficients $a_{11}, a_{12}, a_{21}, a_{22}$ such that $\lim_{t\to\infty} x_1(t) = 0$ and $\lim_{t\to\infty} x_2(t) = 0$.

(b) (10%) Let the initial conditions be equal to zero. For the values $a_{11} = -1$, $a_{12} = 1$, $a_{21} = 0$, $a_{22} = -1$, and

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

calculate the response $y(t) = 2x_1(t) - x_2(t)$. Determine at what time the peak value of y occurs.

(c) (5%) For the values $a_{11} = 0$, $a_{12} = 1$, $a_{21} = -1$, $a_{22} = -2$, and $u(t) = \sin(t)$, calculate the steady-state response of $y(t) = x_1(t) - x_2(t)$.

7. (15%) Consider the region R enclosed by the x-axis, $x = 1$ and $y = x^3$, as illustrated below

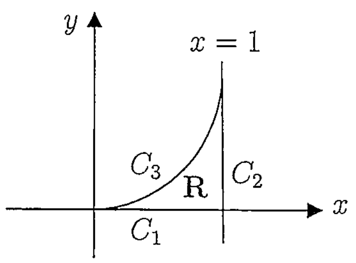

Fig. 1

(a) (3% + 2%) Find the flux of $\overrightarrow{F} = (1 + y^2)\mathrm{j}$ out of R through the two sides C_1 (the horizontal segment) and C_2 (the vertical segment).

(b) (10%) Find the flux of $\overrightarrow{F} = (1 + y^2)\mathrm{j}$ out of the third side C_3.

8. (10%) Consider the following Lyapunov equation

$$XA + A^TX + Q = 0$$

where A is a (n-dimensional) real square matrix, and X, Q are real symmetric matrices.

(a) (5%) Suppose all eigenvalues of A have negative real parts. Show that $X = \int_0^\infty e^{A^T\tau}Qe^{A\tau}d\tau$ is a solution to the Lyapunov equation.

(b) (5%) Suppose Q is positive definite and the Lyapunov equation has a positive definite solution X. Show that all eigenvalues of A have negative real parts.