科目名稱：工程數學乙【電機系碩士班乙組】 題號：431001

※本科目依簡章規定「可以」使用計算機（廠牌、功能不拘） 共2頁第1頁

1. (7%) Find the Laurent series representation of a function

$$f(z) = \frac{5z + j2}{z^2 + jz}$$

with center at $z = j$ in the domain $1 < |z - j| < 2$, $j = \sqrt{-1}$.

2. (8%) Evaluate the following integral:

$$\int_C \frac{z^3 e^{1/z}}{1 + z^3} dz,$$

where C denotes a counterclockwise simple closed contour $|z| = 3$.

3. (15%) Compute the Fourier transform $F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$ of a signum function $f(t)$ defined as

$$f(t) = \begin{cases} -1, & t < 0 \\ 0, & t = 0 \\ 1, & t > 0 \end{cases}$$

Each calculation step is required for obtaining the credit.

4. (20%) 下面的問題共有四個子題，**只要寫出每個子題的答案即可**(不需寫出計算過程)，例如：
(a) $\beta = 1, \gamma = 2$。
Let $\mathrm{a}_1 = \begin{bmatrix} 1 & 0 & 1 & \alpha \end{bmatrix}^T$, $\mathrm{a}_2 = \begin{bmatrix} 1 & \beta & 2 & 2 \end{bmatrix}^T$, and $\mathrm{a}_3 = \begin{bmatrix} -2 & 3 & \gamma & -4 \end{bmatrix}^T$ be three vectors in $\mathbb{R}^4$, where α, β, and γ are three real parameters, and denote $A := \begin{bmatrix} \mathrm{a}_1 & \mathrm{a}_2 & \mathrm{a}_3 \end{bmatrix} \in \mathbb{R}^{4\times 3}$.
   (a) (4%) If $\alpha \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of all positive integers, then find positive integers β and γ such that $\{\mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3\}$ is a linearly dependent set.
   (b) (4%) If $\alpha \notin \mathbb{N}$, find real β and γ such that $\{\mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3\}$ is a linearly dependent set.
   (c) (6%) Now let $\alpha = 2, \beta = -1, \gamma = -5$, and let x be a nonzero vector in the null space $N(A)$ of A. Find the value of k to satisfy $\|\mathrm{x}\|_1 + 2\|\mathrm{x}\|_\infty + k\|\mathrm{x}\|_2 = 0$.
   (d) (6%) Now let $\alpha = 2, \beta = -1, \gamma = -5$, and let d denote the distance between vector $\begin{bmatrix} 1 & 4 & 0 \end{bmatrix}^T$ and $R(A^T)$, the range space of A^T. Compute the value of d.

5. (25%) 下面的問題共有四個子題，**只要寫出每個子題的答案即可**(不需寫出計算過程)，例如：
(a) $\|x\| = 1,\ \theta = 30°$或$\theta = \pi/6$。
Consider the inner product space $C[0,1]$ with $\langle f, g\rangle := \int_0^1 f(x)g(x)dx$ and the norm $\|f\| := \sqrt{\langle f, f\rangle}$.
Denote $S := \mathrm{span}\{1, x\}$ as a subspace of $C[0,1]$.
   (a) (3%+4%) Compute $\|x\|$ and the angle θ, taken value in $[0, \pi/2]$, between 1 and x.
   (b) (6%) Find a vector $u(x)$ in $C[0,1]$, so that $\{1, u(x)\}$ forms an orthonormal basis for S.
   (c) (6%) Find the vector $p(x)$ in S that is closest to $\sqrt{x}$ on $[0,1]$.
   (d) (6%) Let $q(x)$ be the vector in $S^\perp$ that is closest to $\sqrt{x}$. Compute $\|q(x)\|^2$.

6. (20%) Consider the following system of differential equations:

$$\dot{x}_1(t) = a_{11}x_1(t) + a_{12}x_2(t)$$
$$\dot{x}_2(t) = a_{21}x_1(t) + a_{22}x_2(t) + u(t)$$

where $a_{11}, a_{12}, a_{21}, a_{22}$ are constant coefficients.

(a) (5%) Suppose $u \equiv 0$ and the equations are driven by non-zero initial conditions. Determine the conditions on the coefficients $a_{11}, a_{12}, a_{21}, a_{22}$ such that $\lim_{t\to\infty} x_1(t) = 0$ and $\lim_{t\to\infty} x_2(t) = 0$.

(b) (10%) Let the initial conditions be equal to zero. For the values $a_{11} = -1$, $a_{12} = 1$, $a_{21} = 0$, $a_{22} = -1$, and

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

calculate the response $y(t) = 2x_1(t) - x_2(t)$. Determine at what time the peak value of y occurs.

(c) (5%) For the values $a_{11} = 0$, $a_{12} = 1$, $a_{21} = -1$, $a_{22} = -2$, and $u(t) = \sin(t)$, calculate the steady-state response of $y(t) = x_1(t) - x_2(t)$.

7. (5%) Prove the following statements

(a) (3%) The Laplace transform is a linear operation.

(b) (2%) Suppose the Laplace transform of a function $y(t)$ is equal to $Y(s)$. Then the Laplace transform of $y(t-a)$ is equal to $\mathrm{e}^{-sa}Y(s)$.