科目名稱：高等微積分【應數系碩士班丙組】
※本科目依簡章規定「不可以」使用計算機

題號：424003

Question 1 (10 marks)
Evaluate the limit:

$$\lim_{n\to\infty} \frac{\pi + \sqrt{\pi} + \cdots + \sqrt[n]{\pi}}{n}.$$

[You need to give convincing reasons when evaluating this limit.]

Question 2 (10 marks)
Prove that the series

$$\sum_{n=1}^{\infty} \sin\left(n\pi + \frac{1}{\ln(n+1)}\right)$$

is conditionally convergent.

Question 3 (15 marks)
Prove that the sequence of functions

$$f_n(x) = x^n(1-\cos 2\pi x),\ \ n \ge 1$$

is uniformly convergent over the interval $[0,1]$.

Question 4 (15 marks)
Prove that the series

$$\sum_{n=1}^{\infty} xe^{-nx}$$

is uniformly convergent for $x \ge \delta$ for each fixed $\delta > 0$, but fails to be uniformly convergent for $x > 0$.

Question 5 (12 marks)
Use the Implicit Function Theorem to prove that the equation

$$x^2 + y + \cos\left(x + \frac{3}{2}\pi e^y\right) = 0$$

uniquely defines y as a function of x near the point $(0,0)$. Moreover, find the derivative $\frac{dy}{dx}$ at $x = 0$.

Question 6 (14 marks)
Prove that the directional derivative of the function

$$f(x,y) = \frac{x^2 y}{x^4 + y^2} \quad \text{if } (x,y) \ne (0,0) \quad \text{and} \quad f(0,0) = 0$$

exists at $(0,0)$ in every direction, but f is not continuous at $(0,0)$.

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Question 7 (12 marks)

Evaluate the surface integral

$$\iint_{\Sigma}(xy+yz+zx)dS$$

where Σ is the portion of the cone $z=\sqrt{x^2+y^2}$ cut by the cylinder $x^2+y^2=2x$.

Question 8 (12 marks)

Evaluate the triple integral

$$\iiint_{\Omega} xyzdxdydz$$

where Ω is the region enclosed by the unit sphere $x^2+y^2+z^2=1$ and the coordinate planes $x=0, y=0, z=0$.

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