國立中山大學 102 學年度碩士暨碩士專班招生考試試題

科目名稱：機率論【應數系碩士班甲組】　　題號：424007

※本科目依簡章規定「不可以」使用計算機　　共 1 頁第 1 頁

(1) A random number N of dice is thrown. Let A_i be the event that $N = i$, and assume that $P(A_i) = 2^{-i}, i \geq 1$. The sum of the score is S.

(a) Find the conditional probability that $N = 2$ given $S = 4$. (10pts)

(b) Find the conditional probability that $S = 4$ given N is even. (10pts)

(2) Let X_1, X_2, X_3 be independent random variables taking values in the positive integers and having mass functions given by $P(X_i = x) = (1-p_i)p_i^{x-1}$ for $x = 1, 2, \ldots$, and $i = 1, 2, 3$. Show that

$$P(X_1 < X_2 < X_3) = \frac{(1-p_1)(1-p_2)p_2p_3^2}{(1-p_2p_3)(1-p_1p_2p_3)}. (10pts)$$

(3) Suppose X and Y are independent r.v.'s, with $X \sim \text{Gamma}(\alpha_1, \lambda)$, and $Y \sim \text{Gamma}(\alpha_2, \lambda)$. Find $E(X \mid Z)$, where $Z = X + Y$.(10pts)

(4) Let X and Y be independent random variables each having the uniform distribution on $[0, 1]$. Let $U = \min\{X, Y\}$ and $V = \max\{X, Y\}$.

(a) Find $E(U)$. (10pts).

(b) Find $cov(U, V)$. (10pts)

(5) Let X, Y, Z be independent and exponential random variables with respective parameters λ, μ, ν. Find $P(X < Y < Z)$.(10 pts)

(6) Let X and Y have the bivariate normal density function

$$f(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} exp\{-\frac{1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)\}.$$

(a) Show that X and $Z = (Y - \rho X)/\sqrt{1-\rho^2}$ are independent $N(0,1)$ variables. (10pts)

(b) Show that $P(X > 0, Y > 0) = \frac{1}{4} + \frac{1}{2\pi}\sin^{-1}\rho$. (10pts)

(7) Let X have the binomial distribution with parameters n and p, and show that

$$E(\frac{1}{1+X}) = \frac{1-(1-p)^{n+1}}{(n+1)p},$$

and find the limit of this expression as $n \to \infty$ and $p \to 0$.(10pts)