考試科目	統計學A	所別	金融系/金融管理組	考試時間	2月23日(六)第二節

Instructions. (i), Two question sheets in total. (ii), Answer **ALL** questions.

1. For a random sample of size n from a population of size N, consider the following as an estimate of the sample mean μ :

$$\bar{X}_c = \sum_{i=1}^{n} c_i X_i$$

where the c_i are fixed numbers and $X_1, \cdots, X_n$ is an i.d.d. sample.

   (a) (5 points) Find a condition of the c_i such that the estimate is unbiased.

   (b) (10 points) Show that the choice of c_i that minimizes the variance of the estimate subject to this condition is: $c_i = \frac{1}{n}$, where $i = 1, \ldots, n$.

2. (a) (10 points) Let X be a normally distributed random variable, i.e. $X \sim N(\mu, \sigma^2)$. Show that:

$$E\left[e^{\alpha X} 1_{l \le X \le h}\right] = e^{\alpha\mu + \frac{1}{2}\alpha^2\sigma^2}\left\{N\left(\frac{h-(\mu+\alpha\sigma^2)}{\sigma}\right) - N\left(\frac{l-(\mu+\alpha\sigma^2)}{\sigma}\right)\right\}$$

   where $1_{l \le X \le h} = 1$ for $l \le X \le h$, and $1_{l \le X \le h} = 0$ otherwise; and α, l, h are some arbitrary constants.

   (b) (5 points) If $Y = e^X$, then the random variable Y is said to be lognormally distributed. Find the mean, μ_Y , and variance, σ_y, of Y.

   (c) (5 points) Based on the results obtained above, find :

$$E\left[(Y-K)\,1_{Y>K}\right]$$

   for some arbitrary constant K.

3. OLS (ordinary least-squares method) as a technique in fitting the best straight line $Y_i = b_0 + b_1 X_i$ to a sample of X and Y observations involves minimizing the sum of squared (vertical) deviations of points from the line:

$$\min \sum \left(Y_i - \hat{Y}_i\right)^2$$

where Y_i refers to the actual observations, and $\hat{Y}_i$ refers to the corresponding fitted values, so that $Y_i - \hat{Y}_i = e_i$ are the residuals.

   (a) (5 points) Derive the following normal equations:

$$\sum Y_i = n b_0 + \hat{b}_1 \sum X_i$$

$$\sum X_i Y_i = \hat{b}_0 \sum X_i + \hat{b}_1 \sum X_i^2$$

   where n is the number of observations and $\hat{b}_0$ and $\hat{b}_1$ are estimators of the true parameters b_0 and b_1.

   (b) (5 points) Hence, solve for $\hat{b}_1$ and $\hat{b}_0$.

   (c) (5 points) What is meant by an unbiased estimator? How is bias defined?

   (d) (5 points) What is meant by the best linear unbiased estimator (BLUE)? Why is this important?

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4. The Pareto distribution has been widely used as a model in economics for a decaying function with a slowly decaying tail:

$$f\left(x|x_0,\theta\right)=\theta x_0^{\theta}x^{-\theta-1},\ x\geq x_0,\ \theta>1$$

Assume that $x_0>0$ is given and that $X_1,\cdots,X_n$ is an i.d.d. sample.

(a) (5 points) Find the MME (method of moments estimate) of θ.

(b) (10 points) Find the MLE (maximum likelihood estimator) of θ.

(c) (5 points) Find the asymptotic variance of the MLE.

(d) (5 points) Find a sufficient statistic for θ.

5. Let X be a binomial random variable with n trials and probability p of sucess.

(a) (5 points) What is the generalized likelihood ratio for testing

$$H_0: p=0.5 \text{ against } H_A: p\neq 0.5$$

(b) (5 points) Show that the test rejects for large values of $|X-n/2|$.

(c) (5 points) Using the null distribution of X, show how the significance level corresponing to a rejection region $|X-n/2|>k$ can be determined.

(d) (5 points) If $n=10$ and $k=2$, what is the significance level of the test?

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