(20%) **1.** Let $X_1, ..., X_n$ be a rndom sample drawn from the normal distribution $N(1, \sigma^2)$.

(10%) (a) Please provide a consistent estimator for σ^2. You need to verify your answer. Can you also give a consistent estimator of σ?

(10%) (b) Is there UMP test for hypotheses $H_0 : \sigma = 1$ vs. $H_1 : \sigma \neq 1$? You need verify your answer.

(30%) **2.** Let $X_1, ..., X_n$ be a rndom sample drawn from an exponential distribution with pdf

$$f(x, \theta) = \frac{1}{\theta} e^{-x/\theta}, x > 0,$$

for some $\theta > 0$.

(5%) (a) Let $\tau = \frac{1}{\theta}$. Please derive the mle $\hat{\tau}$ of parameter τ.

(15%) (b) Is $\hat{\tau}$ unbiased for τ? Please verify your answer. If $\hat{\tau}$ is not unbiased. Please derive an unbiased estimator.

(10%) (c) Please derive an approximate $100(1-\alpha)\%$ confidence interval for τ based on the central limit theorem.

(20%) **3.** (a) Suppose that we have a random sample $X_1, ..., X_n$ from a distribution with mean μ and variance σ^2.

(5%) (a_1) Suppose that the interest is the minimum variance estimator of parameter $\theta = \sigma^2 + \mu^2$ and you have choices from the following estimators:

$$\hat{\theta}_1 = 3, \hat{\theta}_2 = S^2 + \bar{X}^2, \hat{\theta}_3 = X_1^2,$$

where $S^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2$ and $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$. Which one is your choice of minimum variance estimator? Please state your reason for the choice.

(5%) (a_2) On the other hand, suppose that the interest is an unbiased estimator of θ. Which one is your choice? Please state your reason for the choice.

(10%) (b) Suppose that we have a random variable X with distribution function

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ x^3 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x \ge 1 \end{cases}.$$

We also have observations $x_1, ..., x_n$ of a random sample from uniform distribution $U(0,1)$. Can you give observations of a ramdom sample from the distribution of X?

(30%) **4.** Let $X_1, ..., X_n$ be a random sample drawn from a binomial distribution with pdf $f(x,p) = \binom{m}{x} p^x (1-p)^{m-x}, x = 0, 1, ..., m$.

(10%) (a) Please derive the mle of parameter p. Is it unbiased?

(10%) (b) Please derive the Cramer-Rao lower bound for p. Is the mle the UMVUE?

(10%) (c) Suppose that parameter p has a prior distribution $U(0,1)$. Please derive the posterior distribution of p given $X_1 = x_1, ..., X_n = x_n$ and the posterior mean $E(p|x_1, ..., x_n)$.