1. Please solve the following differential equations. (5 points for each one)

A. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x \tan x$

B. $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = x^4 e^x$

C. $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 1 = 0$

D. $\begin{cases} \frac{dx}{dt} + 2x + 6\int_0^t y\,d\tau = -2 \\ \frac{dx}{dt} + \frac{dy}{dt} + y = 0 \end{cases}$ with $x(0) = -5$ and $y(0) = 6$

2. If y_1, y_2, and y_3 are the linearly independent complementary solutions for a third-order linear differential equation, the particular solution is assumed to be $y_p = u_1y_1 + u_2y_2 + u_3y_3$.

A. Please derive the computation equations for u_1, u_2, and u_3. (12 points)

B. Please use the above derived formulas to find the complete solution for $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} = x$. (8 points)

3. If $\rho(x,y)$ is the length density of a wire (mass per unit length), $m = \int_c \rho(x,y)ds$ is the mass of the wire. Find the mass of a wire having the shape of the semicircle $x = 1 + \cos t$, $y = \sin t$ and $0 \le t \le \pi$, if the density at a point P is directly proportional to the distance from the y-axis. (10 points)

4. Please solve the heat conduction equation in spherical coordinate: (10 points)

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\right), \text{ for } 0 < r < c \text{ and } t > 0 \text{ with } \begin{cases} u(r,0) = r, & 0 < r < c \\ t > 0, & u(c,t) = 5 \end{cases}$$

5. Find the solution $u(r,\theta)$ for a concentric circle plate as: (10 points)

$$\frac{\partial u}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right), \text{ for } 0 < r < 2 \text{ and } t > 0 \text{ with } \begin{cases} u(r,0) = \begin{cases} 200, & 0 < r < 1 \\ 100, & 1 < r < 2 \end{cases} \\ t > 0, \quad u(2,t) = 100 \end{cases}$$

6. Crank-Nicholson method is used to solve the partial differential equation $\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$ with the following conditions: $\begin{cases} u(x,0) = 3x + 1, \quad 0 < x < 1 \\ t > 0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \quad u(1,t) = t - 2 \end{cases}$. Please derive the matrices A and B if AU=B and U is the unknown column matrix of u_0, u_1, u_2, and u_3. (That is, 4 equal intervals.) (15 points)

7. The Dufort-Frankel method for the partial differential equation $\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$ is

$$\frac{T_i^{n+1} - T_i^{n-1}}{2\Delta t} = \frac{T_{i+1}^{\ n} - (T_i^{n+1} + T_i^{n-1}) + T_{i-1}^{\ n}}{\Delta x^2}$$, please derive the conditions for consistency. (15 points)