1. (15%) Let $X_1, X_2, X_3, \cdots$ be a sequence of independent and identical random variables, each with expectation μ and variance σ^2. Prove that the distribution $\dfrac{X_1 + X_2 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}$ converges to the distribution of a standard normal random variable.

2. (10%) Dr. Windler's secretary accidentally threw a patient's file into the wastebasket. A few minutes later, the janitor cleaned the entire clinic, dumped the wastebasket containing the patient's file randomly into one of the seven garbage cans outside the clinic, and left. Determine the expected number of cans that Dr. Windler should empty to find the file.

3. (10%) Let X, Y, and Z be three independent Poisson random variables with parameters λ_1, λ_2, λ_3, respectivley. For $y = 0, 1, 2, \ldots, t$, calculate $P(Y = y \,|\, X + Y + Z = t)$.

4. (15%) Let X be a gamma random variable with parameters r and λ. Derive a formula for the moment generating function $M_X(t)$, and use it to calculate the mean $E(X)$ and the variance $Var(X)$.

5. (20%) Determine the following statements are true (T) or false (F). (Need not to state reasons.)

   (a) For two square matrices A and B, if $AB = I$, the identity matrix, then $BA = I$.

   (b) For a negative-definite matrix M, its determinant $\det(M) < 0$.

   (c) The set of all 5×4 matrices is a vector space.

   (d) For a system of linear equations, $A\mathbf{x} = \mathbf{b}$, if it has no exact solution, then it has a unique least-squares solution.

6. (20%) Consider a matrix A defined by its eigen-decomposition (or diagonalization) as follows,

$$A = EDE^{-1} = \begin{bmatrix} 5 & 4 & 0 & 0 \\ 6 & 5 & 0 & 0 \\ 0 & 0 & 7 & 5 \\ 0 & 0 & 4 & 3 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 4 & 0 & 0 \\ 6 & 5 & 0 & 0 \\ 0 & 0 & 7 & 5 \\ 0 & 0 & 4 & 3 \end{bmatrix}^{-1},$$

   where the columns of matrix E are the eigenvectors of A. Find the eigenvalues and the corresponding eigenvectors of A^T.

7. (10%) Let M be an $n \times n$ real matrix with orthogonal columns, and the Euclidean norm of each column is 1. Find the determinant of M.