Part I. Linear Algebra (50%)

單選題 (每題 5%)

1. The vectors $[1\ 1\ 1\ 1]^T$, $[1\ 0\ -1\ 0]^T$, $[1\ -2\ -3\ 4]^T \in R^4$ are
   (A) linearly independent.
   (B) linearly dependent.
   (C) mutually orthogonal.
   (D) able to form a basis for R^4.
   (E) able to form a spanning set for R^4.

2. If $A = \begin{bmatrix} 4 & 2 & 5 \\ -1 & 0 & 5 \\ 2 & 1 & 2 \end{bmatrix}$, then
   (A) $\text{rank}(A) = 2$.
   (B) $\det(A) = 1$.
   (C) $[2\ 1\ -2]^T$ is in the nullspace of A.
   (D) If $\mathbf{b} = [0\ 1\ 0]^T$, the linear system $A\mathbf{x} = \mathbf{b}$ is consistent.
   (E) A is row equivalent to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

3. $A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}$. Which of the following is **NOT** true?
   (A) 0 is an eigenvalue of e^A.
   (B) e is an eigenvalue of e^A.
   (C) A is diagonalizable.
   (D) e^A is diagonalizable.
   (E) There exists a QR factorization for e^A.

4. Which L is a linear transformation?
   (A) $L(X) = X + \begin{bmatrix} 1 \\ 4 \end{bmatrix}[1\ 1\ 1\ 1]$, $X \in R^{2\text{x}4}$.
   (B) $L(A) = A - A^T$, $A \in R^{\text{nxn}}$.
   (C) $L([x_1\ x_2\ x_3]^T) = x_1 + \sqrt{2}x_2 - x_3$.
   (D) $L(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \begin{bmatrix} x_1 + x_2 \\ 1 \end{bmatrix}$.
   (E) None of the above is linear.

（背面仍有題目,請繼續作答）

5. For any $f, g \in C[-\pi, \pi]$, the inner product in the vector space $C[-\pi, \pi]$ is defined by

$$<f,g>=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)dx.$$

If $(1+\cos x) \in C[-\pi, \pi]$, the length $\|1+\cos x\|$ is

(A) $\sqrt{2}\pi$ (B) $\sqrt{3}$ (C) π (D) $\sqrt{2}$ (E) None of the above is true.

6. Which matrix diagonalizes $\begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix}$?

(A) $\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}$ (B) $\begin{bmatrix}2 & 0\\ 0 & 4\end{bmatrix}$ (C) $\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}$ (D) $\begin{bmatrix}1 & 1/\sqrt{2}\\ 0 & -1/\sqrt{2}\end{bmatrix}$ (E) None of the above is true.

7. Which matrix has trace 1 and determinant -2?

(A) $\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$ (B) $\begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix}$ (C) $\begin{bmatrix}-1 & \sqrt{3}\\ \sqrt{3} & 1\end{bmatrix}$ (D) $\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}$ (E) None of the above is true.

8. Let bases $E=\{\mathbf{v}_1, \mathbf{v}_2\}=\left\{\begin{bmatrix}3\\2\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}$ and $F=\{\mathbf{u}_1, \mathbf{u}_2\}=\left\{\begin{bmatrix}5\\2\end{bmatrix}, \begin{bmatrix}7\\3\end{bmatrix}\right\}$.

What is the transition matrix from E to F?

(A) $\begin{bmatrix}-9 & 16\\ -4 & 7\end{bmatrix}$ (B) $\begin{bmatrix}-5 & -4\\ 4 & 3\end{bmatrix}$ (C) $\begin{bmatrix}-9 & -4\\ 16 & 7\end{bmatrix}$ (D) $\begin{bmatrix}-5 & 4\\ -4 & 3\end{bmatrix}$ (E) None of the above is true.

9. $L(p(x))=\begin{bmatrix}\left.\dfrac{dp(x)}{dx}\right|_{x=1}\\ \dfrac{1}{2}\int_0^2 p(x)dx\end{bmatrix}$. If $p(x)=\alpha x+\beta$, which matrix A satisfies $L(p(x))=A\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$?

(A) $A=\begin{bmatrix}1 & 0\\ 1 & 1/2\end{bmatrix}$ (B) $A=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$ (C) $A=\begin{bmatrix}1 & 0\\ 0 & 4\end{bmatrix}$ (D) $A=\begin{bmatrix}1 & 1/2\\ 1 & 1\end{bmatrix}$ (E) No such matrix exists.

10. Let $\mathbf{x}=[1\ 0\ 3\ 0]^T$, $\mathbf{u}_1=[1\ \ -1\ \ 0\ \ 1]^T/\sqrt{3}$ and $\mathbf{u}_2=[1\ \ 1\ \ 1\ \ 0]^T/\sqrt{3}$. $S=\text{span}\{\mathbf{u}_1, \mathbf{u}_2\}$.

Let T be the orthogonal complement of S, $T=S^{\perp}$. The vector in T that is closest to $\mathbf{x}$ is

(A) $(1/3)[5\ 3\ 4\ 1]^T$ (B) $(1/3)[-2\ -3\ 5\ -1]^T$ (C) $(-2/3)[0\ -2\ -1\ -2]^T$ (D) $(1/6)[3\ -11\ 4\ 5]^T$
(E) None of the above is true.

Part II. Discrete Mathematics 2008 (50%)

1. [10%]

(1) (5%) If statement q has the truth value 1, determine all truth value assignments for the primitive statements, p, r, and s for which the truth value of statement

$$(q \rightarrow [(\neg p \vee r) \wedge \neg s]) \wedge [\neg s \rightarrow (\neg r \wedge q)]$$

is **1**.

(2) (5%) Express the negation of the statement $p \leftrightarrow q$ in term of the connectives $\wedge$ and $\vee$.

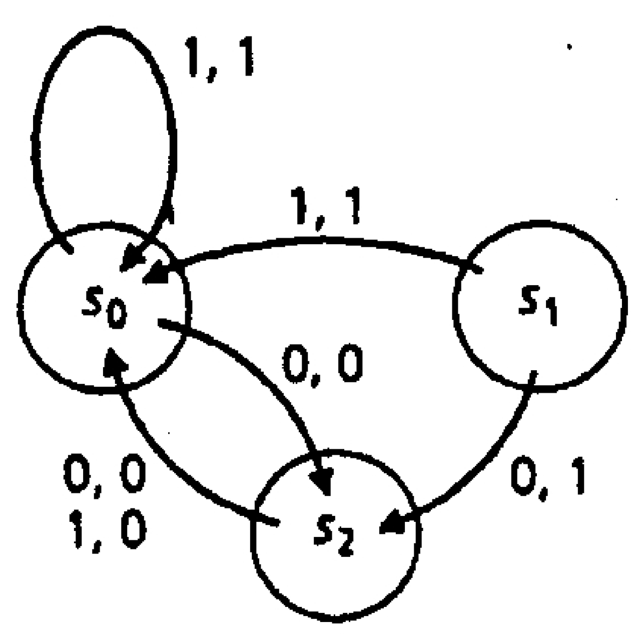
2. [15%]

(1) (5%) Determine which of the following statements are true and which are false.

(a) $Z^+ \subseteq Q^+$
(b) $R^+ \cap C = R^+$
(c) $R^+ \subseteq Q$
(d) $Q^* \cap Z = Z$
(e) $Z^+ \cup R^+ = R^+$

(2) (10%) Prove that for all $n \in Z^+, n > 3 \Rightarrow 2^n < n!$

3. [15%] Let M be the finite state machine in the following figure. For states S_i, S_j, where $0 \leq i, j \leq 2$, let O_{ij} denote the set of all nonempty output strings that M can produce as it goes from state S_i, S_j, e.g., $O_{20} = \{0\}\{1, 00\}^*$. Find O_{22}, O_{11}, and O_{10}.



4. [10%] Find the coefficient of x^{50} in $(x^7 + x^8 + x^9 + \cdots)^6$.