Total: 100 points.

1. (7 points each) Fill in your answers
   - $\int_0^1 \frac{\sin^{-1}x}{\sqrt{1-x^2}}dx =$ (a).
   - The interval of convergence of $\sum_{k=0}^{\infty} k!\,(2\,x-1)^k$ is (b).
   - The volume enclosed by $x=0, y=0, z=0$, and $2\,x+y+3\,z=6$ is (c).
   - Use the Newton-Raphson method to estimate $\sqrt{2}$. The iterative scheme is (d). The approximation with a precision to two decimal places is (e).
   - When the point set (x,y) moves from $(1,2)$ to $(1.05, 2.01)$, the increments dz and Δz for $z = x^2 + 5\,y^2$ are (f) and (g), respectively.

2. (15 points) Show your work to calculate the limit of

$$\frac{\sqrt{2\pi} - \int_{-\infty}^{x} e^{-z^2/2}dz}{x^{-\alpha}e^{-x^2/2}}$$

when $x \to \infty$ for each real value α.

3. (a) (5 points) State the Mean-Value Theorem (single variable).

   (b) (10 points) Let h be a real-valued function and twice differentiable on the real numbers. Prove that for any $a < b$,

$$\frac{\partial}{\partial x}\int_a^b h(x\,z)dz = \int_a^b h'(x\,z)\,z\,dz.$$

4. A differential operator

$$\mathcal{L} = \frac{\partial}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}$$

acting on a real-valued function $P(t,x;\sigma)$ is defined by

$$\mathcal{L}P(t,x;\sigma) = \frac{\partial P}{\partial t}(t,x;\sigma) + \frac{\sigma^2}{2}\frac{\partial^2 P}{\partial x^2}(t,x;\sigma), \qquad (1)$$

where $t \in [0,\infty)$ and $x \in (-\infty,\infty)$ are variables, and $\sigma > 0$ is a parameter. Assume that $P(t,x;\sigma)$ is smooth [that means you can take partial derivatives of any order on P] and $P(t,x;\sigma)$ solves (1) [that means $\mathcal{L}P(t,x;\sigma) = 0$].

   (a) (7 points) Check that for any positive integer n, $\frac{\partial^n}{\partial x^n}P(t,x;\sigma)$ solves (1) as well.

(b) (7 points) For any real number α, $T > t$ and any positive integer n, define

$$Q(t, x; \sigma) = \alpha\,(T-t)\,\frac{\partial^n}{\partial x^n} P(t, x; \sigma).$$

Check that it satisfies

$$\mathcal{L}Q(t, x; \sigma) = -\alpha\,\frac{\partial^n}{\partial x^n} P(t, x; \sigma).$$

(c) (7 points) Prove that

$$\frac{\partial P}{\partial \sigma}(t, x; \sigma) = \sigma\,(T-t)\,\frac{\partial^2 P}{\partial x^2}(t, x; \sigma).$$

[Hint: you may use $\frac{\partial}{\partial \sigma}(\mathcal{L}P) = 0$.]