國立臺灣師範大學九十七學年度碩士班考試入學招生試題

高等微積分 科試題 （數學系 用，本試題共 2 頁）

數學、教學數學組
統計組

注意：
1. 依次序作答，只要標明題號，不必抄題。
2. 答案必須寫在答案卷上，否則不予計分。

1. (10 分) If $S \subset \mathbb{R}$ and $x \in \mathbb{R}$, then x is called an *accumulation point* of S if every 1-ball $B(x;\delta)$ contains at least one point of S distinct from x. Prove that if a bounded subset S of $\mathbb{R}$ contains infinitely many points, then there is at least one point in $\mathbb{R}$ which is an accumulation point of S.

2. (10分) Prove the following version of the Intermediate Value Theorem: Let f be a continuous real-valued function on an interval $[a,b]$ in $\mathbb{R}$, and assume that $f(a)\,f(b) < 0$. Then there is at least one point $c \in (a,b)$ such that $f(c) = 0$.

3. (a) (10 分) Let $f : [a,b] \to \mathbb{R}$ be continuous. Show that f is uniformly continuous on $[a,b]$.

   (b) (5 分) Define $f(x) = 1/x$ for $x \in (0,1)$. Is f uniformly continuous on $(0,1)$? Prove or disprove your answer.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x,y) = \left(e^{x-y} + xy + x(y-1)^4, 1 + x^2 + x^4 + (xy)^5\right)$$

   (a) (8 分) Show that there are neighborhoods U of $(1,1)$ and V of $(2,4)$ such that $f : U \to V$ is one-to-one and onto.

   (b) (7 分) If $f^{-1}(u,v) = \left(g(u,v), h(u,v)\right)$ is the inverse function of f from V to U, find the Jacobian matrix of f^{-1} at $(2,4)$.

5. (10 分) Suppose f is a bounded, Riemann integrable function on the interval $[a,b]$, g is an increasing function on $[a,b]$, and suppose $f \geq 0$. Prove that there is a point $\xi \in [a,b]$ such that

$$\int_a^b f(x)g(x)\,dx = g(a)\int_a^{\xi} f(x)\,dx + g(b)\int_{\xi}^{b} f(x)\,dx.$$

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6. Consider the sequence

$$b_n = \left(1 + \frac{1}{n}\right)^{n+\frac{1}{2}}, \qquad n \in \mathbb{N}.$$

(a) (5 分) Prove that this sequence $\{b_n\}$ converges to a real number L.

(b) (10 分) Which one is larger, b_{2008} or L? Justify your answer.

7. (15 分) Let $f(x)$ be a continuous function in the interval $-1 \le x \le 1$. Prove that:

$$\lim_{h \to 0^+} \int_{-1}^{1} \frac{h}{h^2 + x^2} f(x)\, dx = \pi \cdot f(0).$$

8. (10 分) Let B be the closed disk $x^2 + y^2 \le 1$ on the plane. Evaluate the following double integral

$$\iint_B \ln\sqrt{x^2 + y^2}\; dx\, dy.$$

(試題結束)